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LIPPINCOTT'S ELEMENTARY ALGEBRA

A

SCIENTIFIC AND PRACTICAL TREATISE

BY

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PREFACE.

IT has well been said that the sciences are of a sociable disposition and flourish best in the neighborhood of each other. They are, indeed, in such reciprocal relation that we cannot successfully discover the nature of any one without calling to our aid the helpful truth that abides in a certain number of the rest. A masterful knowledge of arithmetic cannot be obtained by explorations carried on solely within arithmetical limits. It is to Algebra the student must look if he would expand his vision and see clearly both far and wide. However thorough his survey of arithmetical principles may have been, and self-satisfying his attainments in comprehending and applying them, algebra comes to him as a revelation, and opens up before him a domain of truth of which he could have no previous conception. We mean to say, in a word, that the accomplished arithmetician must of necessity be a skilled algebraist as well.

Algebra is, in fact, an enduring corner-stone of mathematical science in general, and for this reason must continually grow in popular favor as an essential educational force; constantly increasing in importance and valuation in the higher

schools, and working its way, in elementary form, into the schools of the lower grade. Inspired by this conviction, we present the following pages for popular consideration and acceptance.

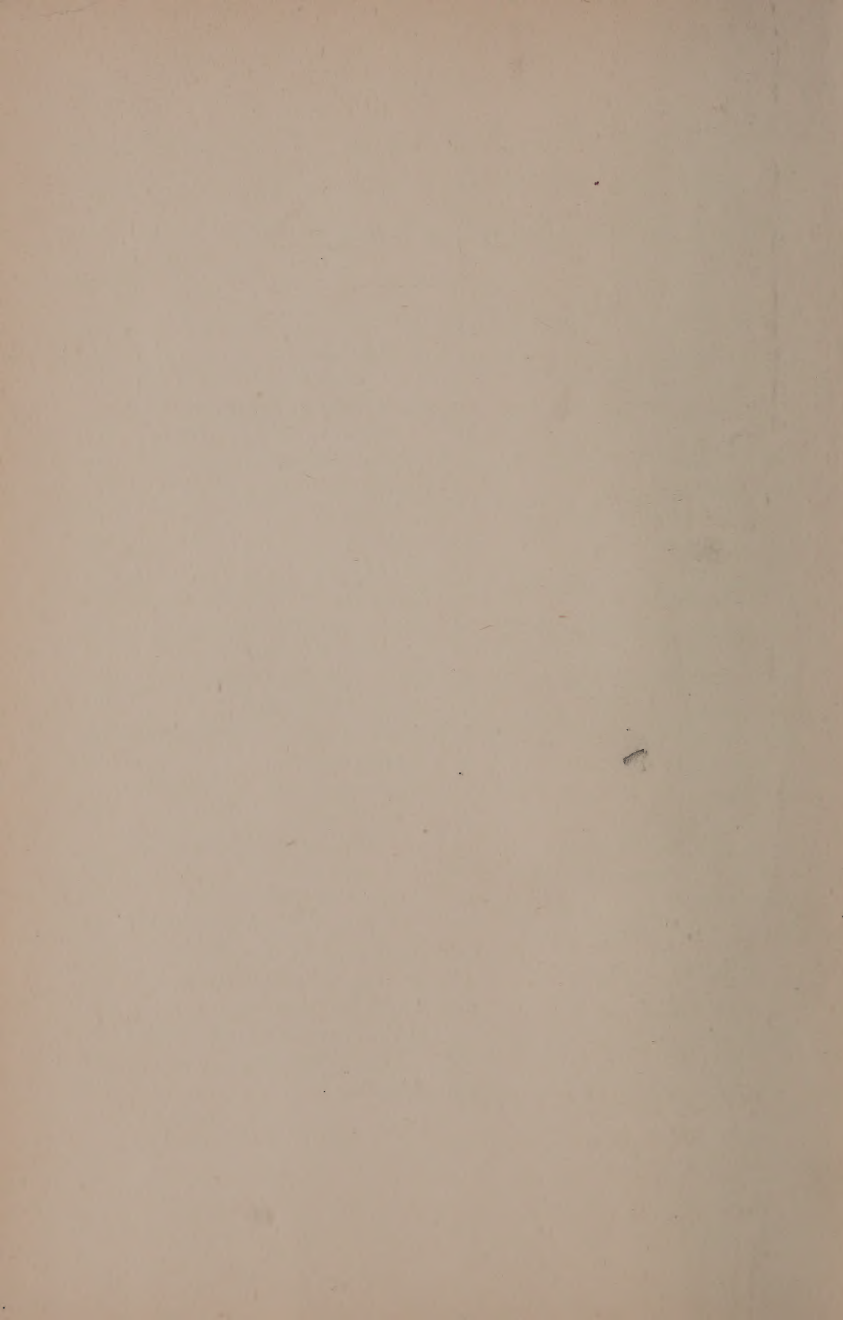
As "'tis in books the chief of all perfections to be plain and brief," we have studied both *plainness* and *brevity*, and we trust there will be found at the same time in our unpretending treatise a fulness of scientific statement and of practical exercises that will meet all ordinary requirements.

The miscellaneous problems and exercises with which our book closes will afford, we believe, a copious supply of varied and interesting matter upon which the teacher can draw to advantage at various stages of his professional work.

J. M. R.

CONTENTS.

	PAGE		PAGE
DEFINITIONS	7	Literal	101
Symbols of Quantity.	8	Special Cases	103
Symbols of Operation	9	EQUATIONS CONTAINING TWO	
Symbols of Relation.	12	UNKNOWN QUANTITIES	118
Algebraic Expression	13	Elimination	119
THE EQUATION.	13	GENERALIZATION	132
ALGEBRAIC USE OF LETTERS.	17	INVOLUTION	135
TRANSPOSITION	19	EVOLUTION	140
Axioms	19	EXPONENTS	153
Principles	20	RADICALS	161
ADDITION AND SUBTRACTION	24	IMAGINARY QUANTITIES	170
SIMPLIFICATION (Parenthesis)	27	QUADRATIC SURDS	173
INSERTION (Parenthesis)	29	QUADRATIC EQUATIONS	177
Review	30	QUADRATIC FORMS	187
MULTIPLICATION	33	SIMULTANEOUS QUADRATIC	
Arrangement	37	EQUATIONS.	190
DIVISION	41	DISCUSSION OF QUADRATIC	
FACTORING	48	EQUATIONS.	197
HIGHEST COMMON FACTOR.	64	Roots as related to Coeffi-	
LOWEST COMMON DIVIDEND	69	cients	200
FRACTIONS	74	RATIO AND PROPORTION	202
INFINITY AND ZERO	96	ARITHMETICAL PROGRESSION	211
Symbol $\frac{0}{0}$	97	GEOMETRICAL PROGRESSION	217
Value of a^0	97	BINOMIAL THEOREM	225
EQUATIONS	98	GENERAL REVIEW	227
Identical	100	MISCELLANEOUS PROBLEMS	
Fractional	100	AND EXERCISES	239
		ANSWERS	301



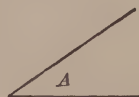
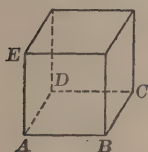
ALGEBRA.

I. Quantity is that which can be measured or estimated ; as distance, weight, time.

Mathematics treats of the exact relations between quantities, and embraces fundamentally Arithmetic, Algebra, and Geometry.

II. Arithmetic treats of quantities or numbers as expressed by the Arabic numerals, 0, 1, 2, 3, 4, etc.

Geometry treats specifically of solids, surfaces, lines, and angles.



III. Algebra, like Arithmetic, treats of quantities or numbers, but, by the use of peculiar symbols, extends its power far beyond the arithmetical limits ; it has been called, not inappropriately, *Universal Arithmetic*.

IV. The Algebraic Symbols are classed as follows :

1. Symbols of Quantity or Number.
2. Symbols of Operation.
3. Symbols of Relation.

Symbols of Quantity.

1. a, b, c, d , etc., representing known quantities—*known* because any value whatever may be given them. a may represent 0, or 1, or any other digit or number.

2. x, y, z, u , etc., representing unknown quantities—*unknown* because their value has to be found. If we say x has the value of a , we cannot know the value of x until we have fixed upon a value for a . If we determine that a represents 10, the value of x has been found, and is 10.

3. $+$ (plus) and $-$ (minus), as marking distinction of character or quality.

Quantity, as algebraically treated, is either *Positive* or *Negative*.

Positive quantities are distinguished by $+$; negative quantities by $-$.

PRINCIPLE.

Positive and negative quantities are diametrically opposite in their signification.

If $+ 10$ denotes dollars *gained*, $- 10$ denotes dollars *lost*. If $+ b$ denotes miles north, $- b$ denotes miles south. b , representing a known quantity, may be assigned a value, say 15; then we have $+ 15$, the miles north, and $- 15$, the miles south.

If height be represented by $+ 7$, and depth by $- 6$, height exceeds depth by 1, and the symbol for the excess is $+ 1$.

If height be represented by $- 7$, and depth by $+ 6$, height again exceeds depth by 1, but the symbol for the excess is $- 1$.

If height be represented by $+ 7$, and depth by $- 7$, there is no excess of height over depth, and the symbol for the condition is 0 (zero).

4. The continuation of a quantity, as c , is represented thus :
 $c \dots\dots$, read "continued indefinitely." $abc \dots\dots$ is
 read " abc and so on."

EXERCISES.

1. What is quantity?
2. Of what does mathematics treat?
3. Embraces what?
4. Arithmetic treats of what?
5. Geometry treats of what?
6. In what respect is algebra like arithmetic?
7. In what respect is it different?
8. What has algebra been called?
9. Name the first class of algebraic symbols.
10. Name the second class.
11. Name the third class.
12. Write five symbols representing known quantities.
13. Select a value for each of them.
14. Write five symbols representing unknown quantities.
15. If the value of y is b , how can you obtain the numerical value of y ?
16. How do $+$ and $-$ differ in signification?
17. If -8 indicates that you owe 8 dollars, what will indicate payment of the debt? What will represent the balance?
18. Write c once, and indicate its continuation.

Symbols of Operation.

1. $+$, sign of *Addition*, used thus : $8 + 10$, $x + y$; read "8 plus 10," " x plus y "; preferably, "the sum, 8 plus 10," "the sum, x plus y ."
2. $-$, sign of *Subtraction*, used thus : $11 - 9$, $c - d$; read

“11 minus 9,” “ c minus d ”; preferably, “the difference, 11 minus 9,” “the difference, c minus d .”

PRINCIPLE.

+ and — indicate operations exactly opposite in kind, effect, and result.

3. $\times$, sign of *Multiplication*, used thus: 8×11 , $a \times b \times c$ (sometimes written $a. b. c$, but commonly abc), read “8 multiplied by 11,” or, “the product of 8 and 11,” “the product of a , b , and c .” Note the use of the period in second line.

8 and 11, and a , b , c , are called *Factors* of their products. Numerical factors precede literal factors, as in the product $6abc$. Here 6 is called the *Coefficient* or Co-factor of abc . $6a$ may be regarded as the coefficient of bc , and $6ab$ as the coefficient of c .

4. $\div$, —, signs of *Division*, used thus: $9 \div 3$, or $\frac{9}{3}$, $a \div b$, or $\frac{a}{b}$; read “9 divided by 3,” or “9 over 3,” “ a divided by b ,” or “ a over b .”

5. 2, 3, 4, etc., as *Exponents*, placed thus: a^2 , a^3 , a^4 , etc., read “ a square,” “ a cube,” “ a fourth power.”

a^2 means the product aa ; a^3 means the product aaa .

a^4 is $aaaa$, in which 4 is an exponent. $4a$ is $a + a + a + a$, in which 4 is a coefficient. Notice carefully the difference.

6. $\sqrt{}$, *Radical Sign*, used thus: $\sqrt{9}$, $\sqrt[3]{8}$, read “square root of 9,” “cube root of 8.”

In $\sqrt[3]{}$, 3 is called the *Index* of the root; in $\sqrt{}$, 2 is the index understood.

EXERCISES.

1. Write algebraically the sum of a , b , and c . Of $2x$, $3y$, $4z$.

2. Write the difference, 7 minus 7, and indicate the difference by a single symbol.

3. How do $+$ and $-$ differ as symbols of quantity? As symbols of operation?

4. If you have travelled continuously 16 miles, and $+7$ signifies your miles north, and -9 your miles south, what symbols will indicate how far and in what direction you are from your starting point?

5. Show by the asterisks that the product of 5×4 is the same as the product of 4×5 .

* * * * *
* * * * *
* * * * *

6. Express accurately and in brief form the product of the factors $x, y, 100, a, b$, and point out the coefficient of y .

* * * * *

7. Why is the product $a \times b$ the same in value as the product $b \times a$?

8. Write the product of x and y "over" the sum of x and y .

9. Write the sum of x, y , and z "over" the difference, e minus f .

10. Indicate the cube of 2, and the cube root of 8.

11. Since 2^3 is $2 \times 2 \times 2$, or 8, and $\sqrt[3]{8}$ is 2, what is the difference in effect between an exponent and an index? What power of 2 is 2^3 ? Write 3 as a coefficient.

12. Write the cube root of 1728, divided by the square root of 144, using symbols.

13. Write 21 as the coefficient of the product $p q r$.

14. Read without ambiguity the following:

1. $a + b$.

4. $\frac{a - b}{a + b}$.

2. $a - b$.

5. $\frac{10 abc}{11x + y}$.

3. $a^2 + b^2 - c$.

6. $\frac{x^4 + c^3}{\sqrt{16}}$.

9. $abc + a^2 + c^2 - b^2$.

7. $2y - 13abxy$.

10. $5a \div 3b + 16 \times 3a$.

8. $\frac{5pqr + 6\sqrt{c}}{4 \times uv}$.

11. $\frac{abcdef}{1 \times 2 \times 3 \times 4}$.

Symbols of Relation.

1. =, sign of *Equality*, used thus: $9 + 5 = 14$, $ax - b = c$, read " $9 + 5$ equals 14," " $ax - b$ equals c ."

2. $>$ and $<$, signs of *Inequality*, used thus: $a + b > c$, $x - y < d$, read "the sum, a plus b , is greater than c ," "the difference, x minus y , is less than d ."

3. $()$, $[\]$, $\{ \}$, --- , signs of *Aggregation*, named respectively *parentheses*, *brackets*, *braces*, *vinculum*, used thus: $6(3 + 2 - 1)$, $6[3 + 2 - 1]$, $6\{3 + 2 - 1\}$, $6 \times \overline{3 + 2 - 1}$, all indicating alike 6 times the value of the quantities enclosed —i. e., 6 times 4.

4. $\therefore$, sign of *Deduction*, used thus: $a = 6$, $b = 5$, $\therefore a + b = 6 + 5 = 11$, read "therefore" or "hence" $a + b = 6 + 5 = 11$.

EXERCISES.

Write the following:

1. The difference, $5a$ minus $3a$, as equal to $2a$.

2. The sum, $5a$ plus $3a$, as equal to $2a$.

3. The difference, $3x$ minus 2, as greater than the sum, $2x$ plus 1.

4. A sign of aggregation, and show by it that x is equal to a times $y + z - c$.

5. Write y times the sum of m and n equal to 6 times the square of p "over" the 4th root of 81.

Algebraic Expression.

1. A combination of algebraic symbols denoting a quantity is called an *Algebraic Expression*. Parts connected by the signs $+$ and $-$ are called *Terms*. $3a + 4bx - 6cy^2$ is an expression consisting of three terms.

2. Terms are called *like* terms when their literal factors are in all respects the same. $6x^2y^3z$ and $7x^2y^3z$ are like terms.

3. A single expression, as $3a$, is called a *Monomial*; an expression consisting of two terms is called a *Binomial*; an expression consisting of three terms is called a *Trinomial*; an expression consisting of two or more terms is called in general a *Polynomial*.

EXERCISES.

Define the following :

- | | |
|--------------------------|----------------|
| 1. Algebraic expression. | 4. Binomial. |
| 2. Terms. | 5. Trinomial. |
| 3. Monomial. | 6. Polynomial. |

Write :

1. A monomial.
2. A binomial (terms *like*).
3. A trinomial (terms *unlike*).
4. A polynomial (four terms).
5. A binomial as equal in value to a trinomial.

The Equation.

1. An equation is the expression of equality between two quantities, made by writing the sign of equality between them. $7 + 8 - 9 = 6$, $2x + 3x = 10$, are equations; read, "The sum of 7 and 8, diminished by 9, equals 6." "Two times x plus 3 times x equals 10."

2. The part of an equation preceding the sign of equality

is called the *First Member*; the part following the sign of equality is called the *Second Member*.

3. The solution of an equation depends upon **Simplification**—a process that consists in removing symbols of operation by performing the operations they indicate or imply.

a. Let the equation be $7 + 8 - 9 = 6$.

The signs of operation are $+$ and $-$.

1. Removing $+$, we have $15 - 9 = 6$.

2. Removing $-$, we have $6 = 6$.

By such a result an equation is said to be *verified*.

b. Let the equation be $5 \times 14 \div 7 - 12 \div 6 \times 2 + 8 = 17$.

The different symbols of operation are $+$, $-$, $\times$, $\div$.

The law or order of their removal is, in this and in all other cases, as follows: $\times$, $\div$, $-$, $+$.

NOTE. $+$ may sometimes be conveniently removed before $-$.

1. Removing $\times$, we have $70 \div 7 - 12 \div 12 + 8 = 17$.

2. Removing $\div$, we have $10 - 1 + 8 = 17$.

3. Removing $-$, we have $9 + 8 = 17$.

4. Removing $+$, we have $17 = 17$, which verifies the equation.

EXERCISES.

Read and verify the following:

1. $3 + 6 \times 2 = 15$.

2. $10 - 3 \times 3 = 1$.

3. $12 \div 3 \times 2 = 2$.

4. $51 \div 3 - 2 \times 6 = 5$.

5. $21 \times 3 \div 7 + 8 \times 3 - 33 = 0$ (see note).

6. $11 \times 20 \div 4 - 12 \div 4 + 6 = 58$.

7. $32 \div 8 \times 4 + 49 \div 7 - 4 = 4$.

$$8. 43 + 9 \div 9 - 4 + 10 - 10 = 40.$$

$$9. 1 \times 2 \times 3 \div 6 - 1 + 100 = 100.$$

$$10. 0 \times 2 \times 4 \times 5 \div 40 + 0 = 0.$$

4. Equations are called :

1. Numerical, when all of the known quantities are represented by *numbers* ; as,

$$3x + 2x = 10, \text{ and } 10 + 6 - 7 = 9.$$

2. Literal, when some or all of the known quantities are represented by *letters* ; as,

$$2ax + b = c, \text{ and } x^2 = ax - c.$$

EXERCISES.

Find by simplification the second member of the following numerical equations :

$$1. 6 + 12 \times 4 =.$$

$$2. 40 - 6 \times 6 =.$$

$$3. 24 \div 6 \times 4 =.$$

$$4. 96 \div 6 - 4 \times 2 =.$$

$$5. 42 \times 6 \div 14 + 16 \times 6 - 66 =.$$

$$6. 22 \times 40 \div 8 - 24 \div 8 + 12 =.$$

$$7. 144 \div 6 \times 8 + 18 \div 6 - 5 =.$$

$$8. 86 + 18 \div 18 - 8 + 20 - 20 =.$$

$$9. 2 \times 4 \times 6 \div 12 - 2 + 200 =.$$

$$10. 0 \times 4 \times 8 \times 10 \div 80 \div 2 + 0 =.$$

5. To indicate that $\div$ is to be removed before $\times$, a parenthesis, $()$, must be used. See (3) above : $24 \div 6 \times 4 = 1$; but $(24 \div 6) \times 4 = 16$. A parenthesis invariably indicates that the signs it includes are the *first* to be removed. A vinculum, _____, may be used instead of $()$; thus $\overline{24 \div 6} \times 4 = 16$, or $\frac{24 \div 6}{4} = 1$.

EXERCISES.

Find the second member of the following equations :

1. $(22 + 7 - 2) \times (12 \div 6) =$.
2. $28 \div 7 - 2 \times 12 \div 6 =$.
3. $22 + (7 - 2) \times (12 \div 6) =$.
4. $13 \times 6 + 4 + 8 \times 2 =$.
5. $13 \times (6 + 4) + 8 \times 2 =$.
6. $4 \times 5 \times 9 - 3 + 9 \div 3 + 6 =$.
7. $4 \times 5 \times (9 - 3) + (9 \div 3 + 6) =$.
8. $(3 + 4) \times 9 - (3 + 6) \div 3 =$.
9. $\overline{3 + 4 \times 9} - \overline{(3 + 6) \div 3} =$.
10. $327 \times 6 \div 109 + 52 \times 5 - \frac{42 \times 5}{2 \times 7}$.

PROBLEMS.

ORAL OR WRITTEN.

1. If A. had 7 marbles and bought twice that number, how many had he then ?

Solution.—1. Let x represent the number A. then had.

2. Hence, the equation is $x = 7 + 7 \times 2$.

3. Removing $\times$, we have $x = 7 + 14$.

4. Removing $+$, we have $x = 21$, the number he then had.

2. B. had 82 dollars ; after paying 3 dollars apiece for 9 books, how many dollars had he left ?

The equation is $x = 82 - 3 \times 9$.

3. C. gave \$6 for one dozen knives ; how many at that rate can be bought for \$1.00 ?

The equation is $x = \$1.00 \div (\$6.00 \div 12)$.

4. D. bought 5 gross of pens for \$3.60 ; at that rate what would 50 pens cost ?

The equation is $x = (\$3.60 \div 720) \times 50$.

5. E. having earned on Monday 76 cents, and on Tuesday 78 cents, spent on Wednesday 9 times 17 cents; how much of his earnings had he left?

6. If 19 horses can be bought for \$950, how many horses can be bought for \$1350?

7. The quotient is 367, the divisor is 445, and the remainder 189. What is the dividend?

8. The gift of \$1.50 per month will amount to \$36 in how many years?

9. If a man earns \$70 per month for 7 months, and \$82 for 5 months, and spends one-third of these earnings for the rent of a house, how much remains for other expenses?

10. Two employés receive \$16 per week each, but expend weekly \$11 and \$8 respectively. How much do they save in 13 weeks?

11. One basket contains 43 apples, another 53 apples, and a third 5 times the difference of these numbers. How many apples in the 3 baskets?

12. What is the cost of 14 hats and 13 caps, if hats are worth \$4 apiece and caps one-half as much?

13. A boy earns half as much as a man; a man earns \$2 a day. What will 9 men and 6 boys earn in 11 days?

14. If 14 sheep cost \$98, for how much must 5 of them be sold to gain \$5 by the sale?

15. If 6 arithmetics, at 40 cents each, and 5 histories together cost \$5, how much do the histories cost apiece?

Algebraic Use of Letters.

1. Numerical factors of a product require the interposition of the sign $\times$; as in $3 \times 5 \times 7$.

2. Literal factors and parentheses are written in a product without the use of $\times$; as in $6abx$, $(a+b)(a-b)$.

3. Although the product of two or more factors is the same in whatever order the factors are arranged, for convenience the numerical factors precede the literal factors; as in $7 \times 6abx$.

4. In the expression $6abx$, 6 is called the coefficient, or co-factor, of abx . $6a$ may be called the coefficient of bx , and $6ab$ the coefficient of x .

EXERCISES.

1. If $a = 0$, what is the value of $5a$?

Ans. 5 times $a = 5$ times 0, which is 0.

2. If $a = 1$, what is the value of $5a$?

3. If $a = 6$, what is the value of $5a + a$?

4. If $a = 11$, what is the value of $7a - 7a$?

5. If $a = 25$, and $b = 50$, what is the value of $2a + b$?

Of $2a - b$? Of $2ab$? Of $2a \div b$? Of $\frac{2a + 2b}{b}$?

6. If $a = 1$, $b = 2$, $c = 3$, $d = 4$, find the numerical value of $5a + 3c - 3b - 2d$.

7. Of $bcd + cda + dab + abc$.

8. Of $(3a + 4d)(2c - 3b)$.

9. Of $\frac{5a}{b+c} - \frac{2b}{c+a} - \frac{2c}{b+a} + \frac{2d}{a+c}$.

10. If $a = 5$, $b = 3$, show that the following expressions are equal to one another:

$$\text{I. } (a-b)(8a-10b).$$

$$\text{II. } (4ab \div 2ab)(20ab \div 10b).$$

$$\text{III. } \frac{12ab}{3a+b} \times \frac{5b+a}{5b-a}.$$

11. Find also the value of $(a^2 + b^2)(a - b)$.

12. Of $\sqrt{5^2 + 3^2 + 2}$.

Transposition.

1. The simplification of an equation—that is, the removal of its symbols of operation—often requires the auxiliary process of **Transposition**, which is the bringing of any term of an equation from one member to the other, without destroying the equality.

This feature of simplification depends upon certain self-evident truths called **Axioms**.

2. Important axioms are :

1. If equals be added to equals, the sums are equal.
2. If equals be subtracted from equals, the remainders are equal.
3. If equals be multiplied by equals, the products are equal.
4. If equals be divided by equals, the quotients are equal.

3. Solve the equation $x - 7 = 6$.

As the terms stand simplification is impossible.

1. Adding 7 to the equals (Ax. 1), we have

$$x - 7 + 7 = 6 + 7.$$

2. -7 cancels $+7$, and we have $x = 6 + 7$.

3. Removing $+$, we have $x = 13$.

In (2) we see that axiom first has transposed 7 from the first member into the second, but has changed its sign.

4. Solve the equation $x + 3 = 7$.

1. Subtracting 3 from the equals (Ax. 2), we have

$$x + 3 - 3 = 7 - 3.$$

2. $+3$ cancels -3 , and we have $x = 7 - 3$.

3. Removing $-$, we have $x = 4$.

In (2) we see that axiom second has transposed 3 from the first member into the second, but has changed its sign from + to -.

PRINCIPLE.

The transposition of a term requires its sign to be changed from + to -, or from - to +.

5. Solve $3x + 2x = 10$, and find the values of $3x$ and $2x$.

1. Removing +, we have $5x = 10$.

2. Removing 5 (Ax. 4), $x = 2$.

3. By Ax. 3, $3x = 6$,
and $2x = 4$.

6. Define the following terms :

Equation.	Literal equation.	Literal factor.
Member.	Parenthesis.	Coefficient.
Simplification.	Vinculum.	Transposition.
Numerical equation.	Numerical factor.	Axiom.

EXERCISES.

Solve the following by the direct use of axioms, and also by transposition :

1. $2x - 3 = 7$.

6. $5x - 5 = 4x - 11$.

2. $2x + 4 = 14$.

7. $4x + 15 = 3 - 2x$.

3. $3x - 8 = 10$.

8. $3x + 15 = x + 25$.

4. $2x - 3 = 8 + x$.

9. $3x + 4 = 5(x - 2)$.

5. $3x + 14 = 2 - x$.

10. $-x + 24 = 3x + 12$.

11. $-32x - 30x + 500 = -58$.

12. $-2x - \frac{1}{2} = \frac{1}{2} - 3x$.

13. $5(x - 3) - 7 - 2 = 24 + 3(8 - x)$.

14. $14x + 6(108 - x) = 1128$.

15. $.375x - 1.875 = .12x + 1.185$.

NOTE.—All the terms of an equation may be transposed—that is, all the signs may be changed. No. 10 may be written $x - 24 = -3x - 12$.

PROBLEMS.

ORAL OR WRITTEN.

1. If a number, represented by x , be added to itself, what will represent the sum?

Solution.— $x + x = 2x$.

2. If the number x be added to twice itself, what will represent the sum? $x + 2x =$ what?

3. If A. has a dollars, and B. three times as many, how many have both?

4. If A. has x dollars, B. twice as many, and both have \$60, how many dollars has each?

Solution.—Since A. has x , B. has $2x$, and both have \$60,

1. The equation is $x + 2x = 60$.

2. Removing +, $3x = 60$.

3. Removing 3 (Ax. 4), $x = 20$, A.'s sum.

$\therefore$ (Ax. 3) $2x = 40$, B.'s sum.

5. If A. has a certain sum of money and B. has 5 times that sum, and both have \$300, how much money has each?

Solution.—Let $x =$ A.'s sum,
then $5x =$ B.'s sum.

1. $\therefore x + 5x = \$300$, the sum both have.

2. Removing +, $6x = 300$.

3. Removing 6 (Ax. 4), $x = 50$, A.'s sum.

$\therefore$ (Ax. 3) $5x = 250$, B.'s sum.

6. A boy has three times as many marbles as his brother; together they have 52. How many has each?

7. The sum of two numbers is 120; the first is 4 times as great as the second. Find the two numbers.

8. Jane, Ann, and Mary have 60 roses. Ann has 3 times as many as Jane, and Mary has twice as many as Ann. How many roses has each girl?

9. In a class of 35 pupils there are 6 times as many girls as boys. How many are there of each?

10. A farmer bought a horse, a cow, and a calf for \$130; the cow cost 3 times as much as the calf, and the horse 3 times as much as the cow. Find the cost of each.

11. Divide the number 144 into two parts such that one part is 12 more than the other.

Solution.—Let x = the less part,

then $x + 12$ = the greater part.

$\therefore$ the equation is $x + x + 12 = 144$.

1. By transposing 12, we have $x + x = 144 - 12$.
2. By removing + and -, we have $2x = 132$.
3. By removing 2 (Ax. 4), we have $x = 66$, the less part.
4. Then $x + 12 = 66 + 12 = 78$, the greater part.

12. The sum of two numbers is 231, and the smaller is 31 less than the larger. What are the numbers?

Solution.—Let x = the larger number,

then $x - 31$ = the smaller number.

$\therefore$ the equation is $x + x - 31 = 231$.

1. By transposing 31, we have $x + x = 231 + 31$.
2. By removing + and -, we have $2x = 262$.
3. By removing 2 (Ax. 4), we have $x = 131$, the larger number.
4. Then $x - 31 = 131 - 31 = 100$, the smaller number.

13. James and John have together \$36, and James has \$12 more than John. How many dollars has each?

Solution.—Let x = John's number of dollars,

then $x + 12$ = James' number.

$\therefore$ the equation is $x + x + 12 = 36$.

1. Transposing 12, we have $x + x = 36 - 12$.
2. Removing + and -, we have $2x = 24$.
3. Removing 2 (Ax. 4), $x = 12$, John's number.
4. Then $x + 12 = 12 + 12 = 24$, James' number.

14. Laura and Miriam together have 59 roses, and Laura has twice as many as Miriam and 5 more. How many roses has each?

Solution.—Let x = Miriam's number ;

then $2x + 5$ = Laura's number.

$\therefore$ the equation is $x + 2x + 5 = 59$.

1. By transposing 5, we have $x + 2x = 59 - 5$.
2. By removing + and -, we have $3x = 54$.
3. By removing 3, we have $x = 18$, Miriam's number.
4. Then $2x + 5 = 36 + 5 = 41$, Laura's number.

15. If from three times a number I take 34 and obtain 32, what is that number?

Suggestion.—Equation is $3x - 34 = 32$.

16. A. and B. together paid \$150 ; if A. paid \$50 more than B., how much did each pay?

17. Two boys solved 48 problems, but one solved 6 less than twice as many as the other. How many did each solve?

18. Find three consecutive numbers whose sum is 18.

19. Pay out to A., B., and C. five hundred dollars, so that B. and C. shall each have twice as much as A.

20. Solve the equation $16x - 8 = 20x - 8 - 5x$.

Suggestion.—Transpose from the second number the terms containing x .

21. What number, diminished by 16, will be equal to $\frac{1}{2}$ itself?

Solution.—Let x = the required number ;

then $x - 16 = \frac{1}{2}x$.

1. By Axiom 3, we have

$$2x - 32 = x$$

2. By transposing, we have

$$2x - x = 32.$$

3. By removing -, we

$$\text{have } x = 32$$

Or

$$\text{By transposing, } x - \frac{1}{2}x = 16.$$

$$\text{By removing -, } \frac{1}{2}x = 16.$$

$$\text{By Axiom 3, } x = 32.$$

22. Divide the number 130 into two parts such that one part shall be $\frac{1}{12}$ of the other.

23. The wind broke off $\frac{3}{7}$ of a flagstaff, and afterward $\frac{2}{7}$ more; the parts broken off measured 35 feet. How high was the flagstaff at first?

24. A father is four times as old as his son, and the difference of their ages is 24 years less than the sum of their ages. Find the age of each.

25. What number is doubled by adding to it $\frac{1}{5}$ of itself and $\frac{3}{5}$ of itself and 9?

26. If the sum of two numbers is 42, and one is $\frac{3}{4}$ of the other, what are the numbers?

ADDITION AND SUBTRACTION.

1. Addition is denoted by the sign of operation, $+$.

Subtraction, the inverse of addition, is appropriately denoted by the sign $-$.

2. The processes of addition and subtraction are but special methods of simplification.

$$1. + (\text{pos. } a) = \text{pos. } a = + a.$$

$$2. + (\text{neg. } a) = \text{neg. } a = - a.$$

$$3. - (\text{pos. } a) = \text{neg. } a = - a.$$

$$4. - (\text{neg. } a) = \text{pos. } a = + a.$$

3. Hence, we see from (1) and (2) that in adding a quantity its sign is not changed; but we see from (3) and (4) that in subtracting a quantity its sign is changed.

Why should the operations denoted by $+$ and $-$ yield exactly opposite results?

4. A quantity without a sign placed immediately before it is considered positive.

1. Simplify $4 - (3 + 2)$. $-$ signifies subtraction, and subtraction means change of sign. Hence, by removing $-$, we have $4 - 3 - 2$, or $4 - 5$, or -1 .

If a debt is \$5, and there are but \$4 to pay it with, what does the -1 signify?

2. Simplify $5 - (-3)$.

The sign, $-$, before the parenthesis means subtraction, and subtraction implies change of sign; hence, by removing $-$, we have $5 + 3$, a numerical sum that is, nevertheless, the algebraic difference sought.

We see, then, that algebraic subtraction may give us an arithmetical sum.

3. Simplify $5 + (-3)$.

The sign, $+$, before the parenthesis means addition, and addition means no change of sign; hence, by removing $+$, we have $5 - 3$, a numerical difference that is, nevertheless, the algebraic sum sought.

We see, then, that algebraic addition may give us an arithmetical difference.

Expressing this peculiarity in symbols, we have

$$a + (-b) = a - b, \text{ the algebraic sum.}$$

$$a - (-b) = a + b, \text{ the algebraic difference.}$$

To what is this peculiarity due?

PRINCIPLES.

Addition retains unchanged the signs of the terms to be added.

Subtraction changes the signs of the terms to be subtracted.

Obviously $7 + 1 = 1 + 7$, and in general $a + b = b + a$.

This is called the Commutative Law of Addition.

Again, $a + (b + c) = (a + b) + c$.

This is called the Associative Law.

If we subtract m from m , we have $m - m = 0$. 0 signifies here *no quantity*, and is the absolute 0 of arithmetic.

EXERCISES.

1. Simplify $7 + (-11)$.

Solution.—1. Addition requires no change of sign.

2. By removing +, we have $7 - 11$, or -4 .

2. Find the sum of $-x$ and $-4x$.

Process.

$$\begin{array}{r} -x \\ -4x \\ \hline -5x \end{array}$$

Solution.

1. Addition requires no change of sign.

2. — once x and -4 times x are -5 times x .

3. From $7ab$ take $3ab$.

Process.

$$\begin{array}{r} 7ab \\ 3ab \\ \hline 4ab \end{array}$$

Solution.

1. Subtraction changes the sign of the term to be subtracted.

2. Conceiving $3ab$ to be $-3ab$, we say
“ $7ab - 3ab = 4ab$.”

4. Simplify $12x - (-7x)$.

Solution.—1. Subtraction changes the sign of the term to be subtracted.

2. By removing the $-$ from before the parenthesis, we have $12x + 7x = 19x$.

5. From x take $3x$.

6. Find the sum of $-8xy$ and $4xy$.

Process.

7. From $4x^2 - 5x$ take $3x^2 + 2x$.

$$4x^2 - 5x$$

8. Find the sum of $8xy$, $-11xy$, and $3xy$.

$$3x^2 + 2x$$

9. Simplify $3ax - 2ax + 5ax - 3ax$.

10. Simplify $6a - (4a - 2)$.

11. Simplify $3x^3 - 6 - (-x^3 + 2)$.

12. From $3x^3 - 6$ take $-x^3 + 2$.
13. Find the sum of $3x^3 - 6$ and $-x^3 + 2$.
14. From $3 - x + x^2 + x^3 + x^4$ take $4 + x - x^2 - x^3 - 2x^4$.
15. Find the sum of $x^2 + y^2 - 3z^2$, $4x^2 - y^2 + 2z^2$, $z^2 - 2x^2$,
 $x^2 - z^2$.

Process.

$$\begin{array}{r}
 x^2 + y^2 - 3z^2 \\
 4x^2 - y^2 + 2z^2 \\
 - 2x^2 \qquad + z^2 \\
 \hline
 x^2 \qquad \qquad - z^2
 \end{array}
 \qquad \text{Arrange like terms in the same column.}$$

16. Simplify $6x - 3x + 5x$.
17. Simplify $-8m + 5m - 5m$.
18. Simplify $6x - (-3y + 4x)$.
19. From 0 subtract $3x$.
20. Find the sum of 0 and $3x$.
21. Find the sum of:
 1. $8a - 10b - 2c$, $5a - 4b - 3c$, $3a + 4b - 6c$.
 2. $4x^2 - 6x + 2$, $8x^2 + 3x - 4$, $-x^2 - 3x - 3$.
 3. $2x^2 - ax + 2a^2$, $3x^2 + 4ax - 5a^2$, $x^2 + ax + 3a^2$.
 4. $4a^2 - 5ab + b^2$, $2a^2 - ab - 3b^2$, $a^2 - 4ab + 5b^2$.
 5. $3x^2y^3 + x^3y^2$, $-6x^2y^3 + 2x^2y^2 - 3x^3y^2$, $5x^3y^2 - 5x^2y^3 - 7x^2y^2$.
22. Subtract $-3a + 4b$ from $4a - 6b$.
23. Subtract $-3x^2 - 3z^2$ from $-4x^2 + 4y^2$.
24. Subtract $a - b$ from $a + b$, and $a + b$ from $a - b$.
25. Subtract $x^4 - x^3 + x^2 - x + 1$ from $x^4 + x^2 + 1$.

The Simplification of Expressions containing Parentheses.

1. A *Parenthesis* is an algebraic expression enclosed within a sign of aggregation. A parenthesis may enclose a

second parenthesis, and the second a third, and so on. When parentheses are thus involved, their removal requires special care.

EXERCISES.

1. Simplify $a - b - [a - (b - c) - c]$.

Solution.—1. There are two parentheses, one in [] enclosing another in ().

Beginning with the outermost parenthesis.	{	2. Removing - from before [], $a - b - a + (b - c) + c$. 3. Removing + from before (), $a - b - a + b - c + c$. 4. Removing + and -, by uniting like terms, we have 0.
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Or,

Beginning with the innermost parenthesis.	{	2. Removing - from before (), we have $a - b - [a - b + c - c]$. 3. Removing - from before [], we have $a - b - a + b - c + c$. 4. Removing + and -, by uniting like terms, we have, as before, 0.
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2. Simplify :

1. $3x - 7y - (x + 2y) - (4y - 7x)$.
2. $(a + b + c) - (a + b - c) - (a - b - c)$.
3. $x^2 - 2xy + y^2 - (x^2 + xy - y^2) - (-2xy + x^2 + y^2)$.
4. $a^2 - (b^2 - c^2) - b^2 + (c^2 - a^2) + c^2 - (b^2 - a^2)$.
5. $4x - 3y - \{(2x + 4y) + 3x + [y - 9x - (2y - x) + (x - y)]\}$.
6. $m - [n - (m - p)]$.
7. $x - [2x + (x - 1)]$.
8. $3x - \{2y + (5x - \overline{3x + y})\}$.
9. $4a - \{3a + [2a - (a - 1)]\}$.
10. $\{50 - [35 - (10 - x)]\} - x$.
11. $3 + (a - b) - (5 - \overline{a^2 - b + 8})$.
12. $2a - (2b - d) - \{a - b - (2c - 2d)\}$.
13. $a - [b + \{a - (b + a)\}]$.

$$14. 16 - x - [7x - \{8x - (9x - \overline{3x - 6x})\}].$$

$$15. 2x - \{a - (2a - [3a - (4a - [5a - (6a - x)])])\}.$$

3. Simplify the following expressions, and find their numerical value, assuming that $a = 0$, $b = 2$, $c = 3$, $d = 4$.

$$1. a \mp b + [a + (b - c) - c].$$

Solution.—1. $a + b + a + b - c - c$.

$$2. 2a + 2b - 2c.$$

[Assigning the values given above, we have]

$$3. 0 + 4 - 6.$$

$$4. 0 - 2 = -2, \text{ Ans.}$$

$$2. a - \{b + [c - (d - b) + a] - 2b\}.$$

Solution.—1. $a - \{b + [c - d + b + a] - 2b\}.$

$$2. a - \{b + c - d + b + a - 2b\}.$$

$$3. a - b - c + d - b - a + 2b.$$

$$4. -c + d.$$

$$5. -3 + 4 = 1, \text{ Ans.}$$

$$3. 3a - [-4b + (4a - b) - (2a - 5b)].$$

$$4. 4c - [a - (2b - 3c) + c] + [a - (2b - 5c - a)].$$

$$5. a - [b + c - a - (a + b) - c] + (2a - \overline{b + c}).$$

$$6. a - [-\{c + a - (a - b) - c\} + 2a - (b - a)].$$

$$7. a - \{b - [a - (c - b) + \overline{c - a} - (a - b - c) - a] + a\}.$$

$$8. 5a - \{-3a - [3a - (2a - \overline{a - b}) - a] + a\}.$$

$$9. d - \{d + [b - (d + b + c - \overline{a + b + d}) - c]\}.$$

$$10. d - [5b - \{d - (5c - \overline{2c - b - 4b}) + 2d - (d - \overline{2b + c})\}].$$

The Insertion of Parentheses.

The insertion of parentheses reverses the operation of their removal: when inserted after sign $+$, the included signs remain unchanged; but when inserted after sign $-$, the included signs *are all changed*.

1. Taking the expression $a + b + c - d$, make a positive parenthesis of $b + c - d$.

Solution.—As $+$ is to precede the parenthesis, the included signs must *not be changed*; hence, we have

$$a + (b + c - d).$$

2. Taking $a - b - c + d$, make a negative parenthesis of $-b - c + d$.

Solution.—As $-$ is to precede the parenthesis, the included signs must *all be changed*; hence, we have

$$a - (b + c - d).$$

3. Taking $a + b + c - d$, make a negative parenthesis of $b + c - d$.

Solution.—As $-$ is to precede the parenthesis, the included signs must *all be changed*; hence, we have

$$a - (-b - c + d).$$

4. Write each of the following expressions, enclosing the last three terms in a negative parenthesis:

1. $2a - 2b - 2c + d + e - f$.

2. $3x - 2a - y - 4b - z - c$.

3. $a - b + c - 5a - b + 2$.

4. $by + ax + bx + cx + cz - xy$.

5. $x + y - z + u - 4 + 5a$.

REVIEW.

1. Solve the following equations:

1. $x - 3 = 2$.

3. $\frac{1}{2}x = 10$.

2. $2x + 12 = 36$.

4. $\frac{1}{5}x + 5 = -9$.

5. $-4x - 8 = -5x - 8$.

2. Find the sum of:

1. $3x^3 - 4a^2 - 4ax^2 + 5a^2x, 6a^3 + 2x^3 + 3ax^2 + a^2x,$
 $18ax^2 - 10x^3 - 14a^2x, 11x^3 + 8a^2x + 4a^3 - 17ax^2$.

2. $2a + b - (c + d), a + (b - c) - d, a + b - (c - d)$.

3. $3a^2 - 4ab - 2b^2$, $5a^2 + 2ab - b^2$, $3ab - 2b^2 - 3c^2$.

4. $a + ab - cd + f$, $3a + 5ab - 6cd - f$, $-5a - 6ab + 6cd - 7f$, $-a + ab + cd + 4f$.

5. $3a + b + c$, $5a + 2b + 3ac$, $a + c + ac$, $-3a - 9ac - 8b$.

3. From :

Subtract :

1. $6ac - 5ab + c^2$

$3ac + 3ab - 7c$.

2. $5a^3 - 4a^2b + 3b^2c$

$-2a^2 + 3a^2b - 8b^2c$.

3. 0

$a^3 - 2a^2 - a - 2$.

4. $a^3 - x^2$

$a^3 + 2ax + x^2$.

5. $a^3 - 2a^2b - 2ab^2 + b^2$

$5a^3 - 7a^2b + 3ab^2$.

4. Simplify :

1. $3x^2 - 2x + 1 - (x^2 + 2x + 3) - (2x^2 - 6x - 6)$.

2. $3a - \{a + b - [a + b + c - (a + b + c + d)]\}$.

3. $11x - \{7x - [8x - (9x - 6x)]\}$.

4. $a^2 - (b^2 - c^2) - b^2 + (c^2 - a^2) + c^2 - (b^2 - a^2)$.

5. $4x - 3y - \{(2x + 4y) + 3x + [y - 9x - (2y - x) + (x - y)]\}$.

5. Assuming that $a = 10$, $b = 3$, $c = 8$, $d = 4$, find the value of the following expressions :

1. $\{a - (b - c)\} + \{b - (c - a)\} + \{c - (a - b)\}$.

2. $\frac{1}{2}a - \frac{1}{3}b + \frac{1}{4}c - (\frac{1}{3}b - \frac{1}{4}c + \frac{1}{5}a)$.

3. $a - [3a + c - \{4a - (3b - c) + 3b\} - 2a]$.

4. $1 - [a^3 - 2 - (-2a^3 - 3)]$.

5. $a - \{a - [b - (a + b + c - \overline{a + b + d}) - c]\}$.

6. Write the following expressions, introducing in each, a parenthesis at the point indicated.

1. $6a + 9b - 6c + 4d - 3e + 2f$.

2. $b - x - y + 3z - 2b + c$.

$$3. a^6 + 4a^5 - 2a^3 - 4a^2 - a - 1.$$

$$4. -3a - 2b + 6c - 4d - e - 2f.$$

$$5. ax - cy - dz + bz + cd + ax.$$

PROBLEMS.

1. When the subtrahend is $8a + 4b - 5c - x$, and the minuend is $-10a - 4b - 15c + 12x$, what is the remainder?

2. What quantity must be added to $3a - 4b + 5c$ to give $5a + 3b - 3c$?

3. What quantity must be added to $a^2 - b^2$ to give $a^2 + 2ab + b^2$?

4. What quantity must be added to $a^2 + 2ab$ to give $a^2 - b^2$?

5. What quantity must be added to $a^2 + 2ab + b^2$ to give 0?

6. What quantity must be subtracted from 0 to give $x^2 - y^2$?

7. What quantity must be subtracted from $x^2 + ax + y^2$ to give $x^2 - 2ax + y^2$?

8. In the equation $x - p = 0$, what is the value of x ? [Show how Axiom 1 applies.]

9. From the difference between the minuend, $10a^2b + 8ab^2 - 8a^2b^2 - b^3$ and $5a^2b - 6ab^2 - 7a^2b^2$ take the sum of $10a^2b^2 + 15ab^2 + 8a^2b$ and $8a^2b - 5ab^2 + a^2b^2$.

10. A man has four sons, each of whom is three years older than his next younger brother; what will represent the age of the oldest, if x represents the age of the youngest?

11. A poulterer has x ducks and y geese. After selling

9 ducks and buying 11 geese, what expression will represent the number of fowls he then has ?

12. If $b = 2$, and $d = 4$, and $e = 3$, what is the numerical value of $\frac{b^e}{d^e}$?

13. Assume

$$A = a^2 + 3a^2b - 5ac + 7by,$$

$$B = 3a^2b - a^2 + 7by + 5ac,$$

$$C = -7by - 3a^2 + ac - 3a^2b.$$

Find the value of :

1. $A + B + C.$

2. $A + B - C.$

3. $A - B + C.$

MULTIPLICATION.

1. Multiplication is the process by which is found the result of taking one quantity as many times as there are units in another.

2. As we have already seen, $+$ and $-$ indicate diametrically opposite forces, and produce diametrically opposite results.

Since $+ 5$ multiplied by $+ 5 = + 25$,

$+ 5$ multiplied by $- 5 = - 25$.

Since $- 5$ multiplied by $+ 5 = - 25$,

$- 5$ multiplied by $- 5 = + 25$.

To use symbols that will represent any values,

Since $(+ a) \times (+ b) = + ab$,

$(+ a) \times (- b) = - ab$.

Since $(- a) \times (+ b) = - ab$,

$(- a) \times (- b) = + ab$.

3. Hence, we perceive that in finding the product of two quantities,

Like signs give +.

Unlike signs give -.

4. Since $aa = a^2$

and $aaa = a^3$

$$a^2 \times a^3 = aa \times aaa = aaaaa = a^5.$$

In general, we have $a^m \times a^n = a^{m+n}$.

Hence, the base remaining the same, *the sum of the exponents of the factors equals the exponent of the product.*

LAWS.

$ab = ba$. Commutative Law.

$a(bc) = ab(c) = abc$. Associative Law.

$c(a+b)$
 $c(a-b)$ } Distributive Law.

$a^m \times a^n = a^{m+n}$. Law of Exponents.

$(+) \times (+) = +$
 $(-) \times (-) = +$
 $(+) \times (-) = -$
 $(-) \times (+) = -$ } Law of Signs.

0 and - 1 as Factors.

If $a + a + a = a \times 3 = 3a$, then

by making successive subtractions of a we have :

1. $a + a = a \times 2 = 2a$,

2. $a = a \times 1 = 1a$,

3. $0 = a \times 0 = 0a$,

4. $-a = a \times (-1) = -1a$,

5. $-2a = a \times (-2) = -2a$, and so on.

That is:

1. When the factors are positive, the product is positive.

2. When 0 is a factor, the product is 0.

3. When a factor is negative, the product is negative.

Multiplication by Monomials.

Multiply $6a^3b^2x$ by $7ab^3x^4$.

Process.

Explanation.

$6a^3b^2x$	1. The product of the coefficients = 42.
$7ab^3x^4$	2. The product of a^3 and a = a^4 .
$42a^4b^5x^5$	3. The product of b^2 and b^3 = b^5 .
	4. The product of x and x^4 = x^5 .

5. Since like signs give +, the product is positive.

6. Hence $42a^4b^5x^5$ is the entire product, containing every factor of both multiplier and multiplicand.

If the above coefficients were - 6 and - 7, would the coefficient of the product be positive or negative? Why?

EXERCISES.

Find the products of the following :

(1) 9	(2) - 6x	(3) 6xy	(4) 70a ² y ³	(5) 70a ² y ³	(6) 8a
<u>10</u>	<u>- 4</u>	<u>- 10</u>	<u>0</u>	<u>- 1</u>	<u>- 9a</u>
(7) 6ay	(8) - 12ax ²	(9) 14xy	(10) - 7x ² b	(11) 8fg	(12) - 3xyz
<u>- 8ay</u>	<u>- 9ax²</u>	<u>- 4ay</u>	<u>- 5c²d</u>	<u>- f²g³</u>	<u>- 9x²y²z²</u>
(13) 22efx	(14) - 8a ⁿ	(15) 10x ⁿ y ^m	(16) 6y ⁿ⁺¹	(17) a ²ⁿ	(18) ay ²ⁿ⁺¹
<u>- 6e²f</u>	<u>6aⁿ</u>	<u>- 3xⁿy^m</u>	<u>- 6yⁿ⁻¹</u>	<u>a³ⁿ</u>	<u>ay²ⁿ⁻¹</u>

19. Find the product of $6a + 4b$ and $3a^3$.

Process.

Explanation.

$6a + 4b$	1. $4b \times 3a^3 = 12a^3b$, first partial product.
$3a^3$	2. $6a \times 3a^3 = 18a^4$, second partial product.
<u>$18a^4 + 12a^3b$</u>	3. $\therefore 12a^3b + 18a^4$ is the entire product.

20. Find the product of :

1. $3x - 2y$ and $- 3abxy$.

2. $7ax - 4by$ and $2xy$.
3. $4x^2y - xy^2$ and $-6xy$.
4. $4xc^2 - 4m^2n$ and $4an$.
5. $m + m^2 - m^3$ and $-9m^2n$.
6. $3x^n - 4x^{n-1}$ and x^2 .
7. $3a^{2n} - 6a^ny$ and $4a^ny^m$.
8. $7x^{1-n} - 5x^{2-n}$ and $5x^n$.
9. $3x - 2y - 4$ and $5x^2$.
10. $x^3 - 3x^2y + 3xy^2 - y^3$ and -3 .

The Product of Negative Factors.

$(-m) \times (-n) = +mn$, the product of 2 terms.
 and $(-m) \times (-n) \times (-p) = -mnp$, the product of 3 terms.

That is, the product of an *even* number of negative terms is *positive*, and the product of an *odd* number of negative terms is *negative*.

21. Find the product of:

1. $-5mn$ and $-5mn$.
2. $-3ab^2$ and $-3a^2b$.
3. $-abc^2$, a^2b^2c and $-a^2bc$.
4. $-3m^2np^3$, mnp^2 and $-m^2np$.
5. $-\frac{1}{2}xy$, $-\frac{2}{3}cx$, and $-\frac{2}{5}a^2y$.

Multiplication by Polynomials.

1. Multiply $x + 20$ by $x + 3$.

Process.

$$\begin{array}{r}
 x + 20 \\
 x + 3 \\
 \hline
 x^2 + 20x \\
 3x + 60 \\
 \hline
 x^2 + 23x + 60
 \end{array}$$

Explanation.

1. $(x + 20)x = x^2 + 20x$, first partial product.
2. $(x + 20)3 = 3x + 60$, second partial product.
3. The whole product = the sum of the partial products.
4. Adding and combining similar terms, we have $x^2 + 23x + 60$.

2. Multiply :

1. $2x - 5$ by $3x + 2$.
2. $2x - 4$ by $4x + 3$.
3. $3x + 6$ by $2x - 8$.
4. $4x - 5y$ by $5x - 4y$.
5. $8x^2 - 6y^2$ by $5x^2 + 4y^2$.
6. $6xy + 7$ by $7xy - 8$.
7. $5a^2 - b^2d$ by $9a^3d + 3ab^2d^2$.
8. $12x^4y - 8x^2y^3$ by $4x^2 + 3y^2$.
9. $a^2 + 2ab$ by $x + y$.
10. $x^3 + 2x^2y$ by $a - b$.

Arrangement.

We cannot but perceive the convenience of having the similar terms of the partial products fall in the same column—a result that is secured by *arranging* the terms of the multiplicand and of the multiplier according to the ascending or descending powers of the *leading letter*.

1. Multiply $x^2 + 3 - 2x$ by $1 + x^2 - x$.

Explanation.

1. Arranging the terms of multiplicand and multiplier according to the descending powers of x , we have $x^2 - 2x + 3$ to be multiplied by $x^2 - x + 1$.

2. Multiplying the multiplicand in succession by x^2 , $-x$, and $+1$, the like terms fall each in its appropriate column.

3. Adding the three partial products we have the entire product.

Process.

$$\begin{array}{r}
 x^2 - 2x + 3 \\
 x^2 - x + 1 \\
 \hline
 x^4 - 2x^3 + 3x^2 \\
 - \quad x^3 + 2x^2 - 3x \\
 \hline
 x^2 - 2x + 3 \\
 \hline
 x^4 - 3x^3 + 6x^2 - 5x + 3
 \end{array}$$

EXERCISES.

2. Find the product of :

1. $m^2 + n^2 + mn$ and $m - n$.
2. $4ab + 3a^2 + b^2$ and $2a + 5b$.

3. $7ax^5 + x^5 + a^2x^5$ and $x + ax$.
4. $x^2 + y^2 + abx$ and $x^2 - y^2$.
5. $x^2y + xy^2 + x^3 + y^3$ and $x - y$.
6. $x^2 + xy + y^2$ and $y^2 - xy + x^2$.
7. $x^2 + x^3 + 1 + x$ and $x - 1$.
8. $x^2 + x^4 + 1$ and $1 + x^2 - x^4$.
9. $x^3 + 1 - 5x^2$ and $5x + 1 + 2x^3$.
10. $x^2 - x + 1$, $x + 1 + x^2$, and $x^2 + 1 + x^4$.

Algebraic terms having the same number of literal factors are said to be of the *same degree*; and a polynomial whose terms are of the same degree is said to be *homogeneous*. When both multiplicand and multiplier are homogeneous, their product is homogeneous.

$$a^2 + 2ab + b^2 \text{—2d degree and homogeneous}$$

$$a^3 + a^2b + b^3 \text{—3d degree and homogeneous}$$

$$a^5 + 2a^4b + a^3b^2$$

$$a^4b + 2a^3b^2 + a^2b^3$$

$$a^2b^3 + 2ab^4 + b^5$$

$$a^5 + 3a^4b + 3a^3b^2 + 2a^2b^3 + 2ab^4 + b^5 \text{—5th degree and hom.}$$

When the product of two or more homogeneous factors is found and is not homogeneous, an error has occurred in the process of obtaining the product.

When no two terms of partial products are similar, the number of terms of the product equals the number of terms of the multiplicand multiplied by the number of terms of the multiplier.

$$3a^2 - 5b^2 + 3c^2 \quad (3 \text{ terms})$$

$$a^2 - b^3 \quad (2 \text{ terms})$$

$$3a^4 - 5a^2b^2 + 3a^2c^2 - 3a^2b^3 + 5b^5 - 3b^3c^2. \quad (6 \text{ terms.})$$

Multiplication may be indicated by the use of parentheses as introductory to the actual finding of the product.

1. Indicate and find the product of $a + b$ and $a + b$.

Process.

$$(a + b)(a + b) = (a + b)^2 \\ = a^2 + 2ab + b^2$$

Explanation.

Enclosing each binomial in a parenthesis, and placing the parentheses in a line, we have $(a + b)(a + b)$; or, since the two factors are equal, we enclose $a + b$ in a single parenthesis and indicate its square or second power; read, "the square of the sum of a and b ". Simplifying or expanding $(a + b)^2$, we have $a^2 + 2ab + b^2$.

2. Indicate the product of $a + b$ and $a - b$ diminished by the square of b . Also simplify.

3. Indicate the product of $a - b$ and $a - b$ diminished by $a^2 - 2ab$. Simplify.

4. Indicate the product of $a^2 + ab + b^2$ and $a - b$ diminished by $a^3 - b^3$. Simplify.

5. Indicate the product of $a^2 - ab + b^2$ and $a + b$ increased by the sum of a^3 and b^3 . Simplify.

Simplify :

1. $(x + y + z)(x + y - z) - (2xy - z^2)$.
2. $(a + b)a - [(a + b)^2 - b(b - a)]$.
3. $[xz - (x - y)(y + z)] - y[y - (x - z)]$.
4. $(a - 1)(a - 2) - 3a(a + 3) + 2[(a + 2)(a + 1) - 3]$.
5. $(x + y)(x^3 - y^3)[x^2 - y(x - y)]$.

REVIEW.

1. Find the product of the following :

1. $-11x^m$ and $-8x^m$.
2. $4x^my^n$, $-x^ny^nz^2$, and $16y^2z^2$.
3. $m^3 + m^2n + n^2$ and m^3n^3 .
4. a^3b^3 and $a^3 - 3a^2b + 3ab^2 - b^3$.

5. $a - b + c$ and $a + b - c$.
 6. $3x^{m+2}y - 3xy^{n+1}$ and $4x^{m+5}y^2 - 4x^4y^{n+2}$
 7. $a + 1$, $3a - 2$, and $3a^2 - a - 2$.
 8. $m + 1$, $m - 1$, $m + 2$, and $m - 2$.
 9. $6(x + y)^{n+1} + 4(x + y)^n - 2(x + y)^{n-1}$, and $(x + y)^{n+2}$.
 10. $3a^2 - 5bd + cf$ and $-5a^2 + 4bd - 8cf$.
 11. $x^2 + x + 1$, $x^2 - x + 1$, and $x^2 + x - 1$.
 12. $1 + 4x - 10x^2$ and $1 - 6x + 3x^2$.
 13. $a^2 + b^2 + c^2 - bc - ac - ab$ and $a + b + c$.
 14. $a^m + b^p - 2c^n$ and $2a^m - 3b$.
2. Expand the following :
1. $(a + b)^2$.
 2. $(a - b)^2$.
 3. $(a + b)(a - b)$.
 4. $(a^2 + ab + b^2)(a - b)$.
 5. $(a^2 - ab + b^2)(a + b)$.
 6. $(a + b)(a + b)(a + b)$.
 7. $(a - b)(a - b)(a - b)$.
 8. $(a + 1)(a + 1)(a + 1)$.
 9. $(a - 1)(a - 1)(a - 1)$.
 10. $(a + b)(a + b)(a - b)$.
 11. $(a - b)(a - b)(a + b)$.
 12. $(a + b + c)^2$.
 13. $(a - b - c)^2$.

3. Inspect the above results, and from the obvious law of formation write the expansion of :

- | | |
|-----------------------|-------------------------|
| 1. $(x + y)(x - y)$. | 5. $(x - y)^3$. |
| 2. $(x + y)^2$. | 6. $(x - y)^2(x + y)$. |
| 3. $(x - y)^2$. | 7. $(x + y + z)^2$. |
| 4. $(x + y)^3$. | 8. $(x - y - z)^2$. |

4. Simplify the following, substituting 0 for a , 1 for b , and -1 for c :

1. $a - 2(b + 3a) - 3\{b + 2(a - b)\}.$
2. $(a + b + c)(a + b - c) - (2ab - c^2).$
3. $4a - [2a - \{2b(a + c) - 2b(b + c)\}].$
4. $(a + 3)^3 - 3(a + 2)^3 + 3(a + 1)^3 - a^3.$
5. $5a - 7(b - c) - [6a - (3b + 2c) + 4c].$

5. Simplify the following, substituting $a + b$ for c , and making $a = b$:

1. $(b + c)(a + b - c).$
2. $c^2(a + b).$
3. $\frac{a + b + c}{2} - a + \frac{a + b + c}{2} - b + \frac{a + b + c}{2} - c.$
4. $(m + 1)^c(m + 1)^c.$
5. $(x^2 - (a + b)x + ab)(x - c).$

DIVISION.

Division is the inverse of multiplication, and is, therefore, the process of finding one of two factors when their product and the other factor are given.

The product of the two factors is the *Dividend*. (D.)

The given factor is the *Divisor*. (D.)

The required factor is the *Quotient*. (Q.)

Hence,

$$D = D \times Q.$$

$$+ab \div (+a) = +b.$$

$$-ab \div (+a) = -b.$$

$$+ab \div (-a) = -b.$$

$$-ab \div (-a) = +b.$$

That is,

Like signs produce + ; }
 Unlike signs produce -. } Law of Signs.

Again,

$$a^5 \div a^3 = \frac{a \times a \times a \times a \times a}{a \times a \times a} = aa = a^2.$$

That is, when a letter appears at the same time in dividend, divisor, and quotient, its exponent in the quotient equals its exponent in the dividend minus its exponent in the divisor.

$$\frac{a^m}{a^n} = a^{m-n}, \text{ Law of Exponents.}$$

$$\frac{b+c}{a} = \frac{b}{a} + \frac{c}{a}, \text{ Distributive Law.}$$

Division by Monomials.

Divide :

- | | |
|-------------------------------|---------------------------------|
| 1. $4a^6$ by a^4 . | 6. $42x^5$ by $-6x^5$. |
| 2. $-36x^7$ by $6x^5$. | 7. $56a^8b^6$ by $-14a^2b^4$. |
| 3. $-70y^9$ by $7y^7$. | 8. $-50z^6b^4$ by $-10z^2b^2$. |
| 4. $-12z^3$ by $4z$. | 9. $-48m^8n^6$ by $-8m^4n^4$. |
| 5. $40x^9y^4$ by $-5x^2y^3$. | 10. $-35p^4q^4$ by $7p^3q^3$. |

Divide :

2. 1. $-16a^3 - 32a^5$ by $-16a^3$.
2. $54a^5 - 72a^3$ by $18a^3$.
3. $-15a^5 + 10a^7$ by $-5a^5$.
4. $-6x^3y^2 - 2x^3y^3$ by $-2x^2y^2$.
5. $-6x^5z^5 - 12x^3z^3$ by $-6x^3z^3$.
6. $a^2 - ab - ac$ by $-a$.
7. $6a^3 - 12a^2b - 18ab^2$ by $-3a$.
8. $x^7y^4 - x^5y^5 - x^4y^4$ by x^4y^4 .
9. $16x^3 - 8x^2y - 12xy^2$ by $-4x$.
10. $6m^4n^2 - 12m^3n^3 - 30m^2n^4$ by $3m^2n^2$.

Divide :

3. 1. $-m + m^2 + m^3 - m^4$ by $-m$.
2. $0 + x + x^2 - x^3$ by x .

3. $a^m - a^{m+1} + a^{m+2}$ by a^m .
4. $6b^{a+1}c - 4b^{a+2}c^2$ by $2b^{a+1}c$.
5. $a^{-3} + a^{-2}b - 2a^{-1}c - ab^{-1}$ by a^{-1} .
6. $1^5 - 2^4 + 3^3 - 4^2 + 5$ by 1.
7. $x^{n+3} + x^{n+2} + x^{n+1} + x^n$ by x^{n+4} .
8. $13x^{m+3} + 7x^{n+2} - x^n$ by x^2 .
9. $111a^{4n} + 63a^{3n} - 3a^n$ by $3a^n$.
10. $15x^m y^n z^r - 35x^{m+2} y^{2n} z$ by $5x^m y^n z$.

Division by Polynomials.

$a^2 + ab + b^2$ and $a + b$ are the factors of $a^3 + 2a^2b + 2ab^2 + b^3$. Multiplying $a^2 + ab + b^2$ by $a + b$, we find the partial products to be :

1. $a^3 + a^2b + ab^2$, and

2. $a^2b + ab^2 + b^3$.

Hence, taking $a^2 + ab + b^2$ for divisor and $a^3 + 2a^2b + 2ab^2 + b^3$ for dividend, we find the quotient $a + b$ as follows :

Process.

$$\begin{array}{r}
 \text{D} = a^3 + 2a^2b + 2ab^2 + b^3 \quad \begin{array}{l} \text{D} \diagup a^2 + ab + b^2 \\ \text{Q} \diagdown a + b \end{array} \\
 \underline{a^3 + a^2b + ab^2} \\
 a^2b + ab^2 + b^3 \\
 \underline{a^2b + ab^2 + b^3} \\
 0
 \end{array}$$

Explanation.

1. Since a^3 in the D is the product of a^2 in the D and of a in the Q, $a^3 \div a^2 = a$, the first term in the Q.

2. Multiplying the D by a , we have the first partial product, $a^3 + a^2b + ab^2$.

3. Subtracting the first partial product from the D, we have the second partial product, $a^2b + ab^2 + b^3$.

4. Since a^2b in the second partial product $= a^2$ in the D $\times b$ in the Q, $a^2b \div a^2 = b$, the second term in the Q.

5. Multiplying the D by b , we have the second partial product.

6. Subtracting again, we have for remainder 0.

7. Hence, $a + b$ is the exact quotient sought.

Brief Directions are :

1. "Arrange" both the D and the d in like manner.
2. Divide the first term in the D by the first term in the d.
3. Multiply the d by this first term in the q.
4. Subtract the product from the D.
5. Divide the first term in the remainder by the first term in the d.
6. Multiply the d by this second term in q.
7. Subtract as before, and, if necessary, repeat steps 5, 6, and 7.

(5), (6), and (7) must be repeated until the entire quotient is found.

To Apply the Directions :

1. Divide $9ab^2 + a^3 - 9b^3 - 5a^2b$ by $3b^2 + a^2 - 2ab$.

1. Arranging, we have $a^3 - 5a^2b + 9ab^2 - 9b^3$ $\left| \begin{array}{l} a^2 - 2ab + 3b^2 \\ a - 3b. \end{array} \right.$

2. Dividing, we have $\frac{a^3 - 2a^2b + 3ab^2}{a - 3b.}$
 a in q. $- 3a^2b + 6ab^2 - 9b^3$

3. Multiplying, we have $- 3a^2b + 6ab^2 - 9b^3$
 $a^3 - 2a^2b + 3ab^2. \quad 0$

4. Subtracting, we have $- 3a^2b + 6ab^2 - 9b^3$.

5. Dividing again, we have $- 3b$.

6. Multiplying by $- 3b$, we have $- 3a^2b + 6ab^2 - 9b^3$.

7. Subtracting, we have for the remainder 0.

2. Divide $x^6 + 10x - 33$ by $3 + x^2 - 2x$.

(1) $x^6 + 10x - 33$ $\left| \begin{array}{l} x^2 - 2x + 3 \\ x^4 + 2x^3 + x^2 - 4x - 11 \end{array} \right.$

(3) $x^6 - 2x^5 + 3x^4$ $\left| \begin{array}{l} x^2 - 2x + 3 \\ x^4 + 2x^3 + x^2 - 4x - 11 \end{array} \right.$

(4) $2x^5 - 3x^4 + 10x - 33$ (2) (5) (5) (5) (5)

(6) $2x^5 - 4x^4 + 6x^3$

(7) $x^4 - 6x^3 + 10x - 33$

(6) $x^4 - 2x^3 + 3x^2$

(7) $- 4x^3 - 3x^2 + 10x - 33$

(6) $- 4x^3 + 8x^2 - 12x$

(7) $- 11x^2 + 22x - 33$

(6) $- 11x^2 + 22x - 33$

(7) 0

EXERCISES.

Divide.

1. $4x^2 + 12x + 9$ by $2x + 3$.
2. $6x^2 - x - 35$ by $3x + 7$.
3. $x^2 - 9y^2$ by $x + 3y$.
4. $x^2 + 2ax + a^2$ by $a + x$.
5. $a^3 - 3a^2y + 3ay^2 - y^3$ by $a - y$.
6. $6x^4 - 96$ by $3x - 6$.
7. $a^2 - b^2 + c^2 - 2ac$ by $a + b - c$.
8. $8x^2 - 10ax - 3a^2$ by $4x + a$.
9. $144x^4 - 145x^2y^2 + 36y^4$ by $4x + 3y$.
10. $x^6 + y^6$ by $x^2 + y^2$.
11. $x^4 + a^2x^2 + a^4$ by $x^2 - ax + a^2$.
12. $a^3 - 4ab^2 - a - 4ab$ by $a - 2b - 1$.
13. $x^6 + 10x - 33$ by $3 + x^2 - 2x$.
14. $x^3 + y^3 + z^3 - 3xyz$ by $x + y + z$.
15. $x^6 - 6x^4 + 5x^2 - 1$ by $x^3 + 2x^2 - x - 1$.
16. $x^4 - y^4$ by $x - y$.
17. $x^{2n} - y^{2n}$ by $x^n + y^n$.
18. $x^4 + x^{-4} - x^2 - x^{-2}$ by $x - x^{-1}$.
19. $1 - x^3$ by $1 - x^2$.
20. $27a^3 - 8b^3 - 27c^3 - 54abc$ by $3a - 2b - 3c$.

REVIEW.

1. Find the sum of:

1. $a + 2b + 3c, 2a - b - 2c, b - a - c, c - a - b$.
2. $2a - 4c + b, 7a - 3c + m, -9a + 6c + 3ab, + 7am$.
3. $5(a + x), -2(a + x), -3(a + x)$.
4. $4\sqrt{x} + 2a^2x, -\sqrt{x} - a^2x, 4\sqrt{x} - a^2x, -3\sqrt{x}$
5. $(b - a)\sqrt{x}, (2a - b)\sqrt{x}, (a + c)\sqrt{x}$.

2. Simplify the following:

1. $(3a^2x - 13abc + 7a^2) - (9a^2x - 13abc)$.

2. $(51a^3b^2c - 18abc - 14a^2y) - (41a^3b^2c - 27abc - 14a^2y)$.
3. $(10a^m - 15b^n - c^p + 5d^q) - (-9a^m + 2b^n + c^p - 5d^q)$.
4. $(\frac{3}{2}a + 36 + \frac{1}{4}c + \frac{1}{2}bc) - (\frac{5}{2}a - 76 - 3bc + \frac{1}{2}c)$.
5. $(\frac{1}{2}y - \frac{5}{2}a - \frac{3}{4}x + \frac{1}{3}a) - (3y + \frac{11}{4}a - \frac{2}{3}x)$.

3. Remove the parentheses from the following:

1. $1 - [1 - (2 - x)] + [4x - (3 - 6x)] + [4 - (-5 + 6x)]$.
2. $2a - [2a - (b + 2x)] + [b - (2x - 2b)]$.
3. $[(1 + x) - (1 + 2x)] + [(1 - x) + (1 - 2x)] - [(1 - x) - (1 - 2x)]$.
4. $11x - \{7x - [8x - (9x - 6x)]\}$.
5. $4x - 3y - \{(2x + 4y) + 3x + [y - 9x - (2y - x) + (x - y)]\}$.

4. Divide:

1. $x^2 + 2ax + a^2$ by $x + a$.
2. $x^2 - 2ax + a^2$ by $x - a$.
3. $x^2 - a^2$ by $x + a$.
4. $x^2 - a^2$ by $x - a$.
5. $x^3 + a^3$ by $x + a$.
6. $x^3 - a^3$ by $x - a$.
7. $x^4 - a^4$ by $x + a$.
8. $x^4 - a^4$ by $x - a$.
9. $x^4 + a^2x^2 + a^4$ by $x^2 + ax + a^2$.
10. $x^4 + a^2x^2 + a^4$ by $x^2 - ax + a^2$.
11. $x^3 + y^3 + z^3 - 3xyz$ by $x + y + z$.
12. $8x^3 - y^3$ by $2x - y$.

5. Simplify:

1. $(x - 3)(x - 5)(x - 7)$.
2. $(x + 5y)(4x + y)(2x - 3y)$.
3. $(x^4 + 1)(x^2 + 1)(x^2 - 1)$.

4. $(x^4 + 16y^4)(x^2 + 4y^2)(x + 2y)(x - 2y)$.
5. $(x - y)^2(x + y)^2(x^2 + y^2)^2$.
6. $(x^2 + 1)^3(x + 1)^3(x - 1)^3$.
7. $(a^2 - 2ax + 4x^2)(a^2 + 2ax + 4x^2)$.
8. $(9a^2 + 3ax + x^2)(9a^2 - 3ax + x^2)$.
9. $(a^m + b^p - 2c^n)(2a^m - 3b)$.
10. $(a^{2n} + x^{2n})(2a^{2n} - 2x^{2n})$.
11. $(\frac{2}{3}x^2 + \frac{1}{4}y^2)(\frac{2}{3}x^2 - \frac{1}{4}y^2)$.
12. $(1 - x)(1 + x)(1 + x^2)(1 + x^4)$.

6. Divide :

1. $8.4x^4 - 1.6x^3 - 10.3x^2 + 10.2x - 3.9$ by $2.4x^2 + 1.6x + 2.6$.
2. $2x^{10} - .075x^5 + 9.65x^7 - 1.05x^6 - 19.25x^8 + 8.5x^9$
by $2.5x^6 - 3x^5 + .5x^7 - .15x^4$.
3. $-\frac{1}{8}d^5 + d^2 - \frac{41}{24}d^3 + \frac{5}{6}d^4$ by $-\frac{3}{4}d^2 + 2d$.
4. $x^4 - \frac{19x^2}{6} + \frac{x}{3} + \frac{1}{6}$ by $x^2 - 2x + \frac{1}{2}$.
5. $1.2a^4x - 5.494a^3x^2 + 4.8a^2x^3 + 0.9ax^4 - x^5$ by
 $0.6ax - 2x^2$.

7. Verify the following :

1. $a - (c - 2) - \left(\frac{2a + 2b}{a + b} \right) = a - c$.
2. $3x^2 - \{5x - [4 - (x - 1)(2x - 3) - 7x]\} = x^2 - 7x + 1$.
3. $2\{b + a(2 - a)\} - \left(\frac{4a^2 - 2a + 4ab}{2a} \right) = 2a - 2a^2 + 1$.
4. $3x + \left(\frac{x + bcx + 4ax}{x} \right) - \left(\frac{4bc}{4} \right) - \left(\frac{8a^2 + 2a}{2a} \right) = 3x$.
5. $-a[a\{(a - b) - (a + b)\} - \overline{a - 2ab}] + a^2 = +2a^2$.

FACTORING.

Formulæ.

1. **Factoring** is the process of finding the factors of a quantity ; it reverses the process of expanding.

2. A **Formula** is the algebraic expression of a rule or a principle.

3. The following formulæ are of vast importance in their relation to factoring, and, in general, to the simplifying of algebraic expressions ; *they should be most thoroughly learned.*

$$1. (a + b)(a + b) = (a + b)^2 = a^2 + 2ab + b^2 ; \text{ hence,} \\ a^2 + 2ab + b^2 = (a + b)^2 = (a + b)(a + b) \text{ as factors.}$$

$$2. (a - b)(a - b) = (a - b)^2 = a^2 - 2ab + b^2 ; \text{ hence,} \\ a^2 - 2ab + b^2 = (a - b)^2 = (a - b)(a - b) \text{ as factors.}$$

$$3. (a + b)(a - b) = a^2 - b^2 ; \text{ hence,} \\ a^2 - b^2 = (a + b)(a - b) \text{ as factors.}$$

EXERCISES.

1. Write and expand the square of $x + y$.

Process.

By formula 1,

$$(x + y)^2 = x^2 + 2xy + y^2$$

Observe that here all the terms are positive, and that $2xy$ is twice the product of $\sqrt{x^2} (= x)$ and $\sqrt{y^2} (= y)$.

2. Write and expand the square of $m - n$.

Process.

By formula 2,

$$(m - n)^2 = m^2 - 2mn + n^2$$

Observe here that $2mn$ is negative.

3. Expand $(p + q)(p - q)$.

Process.

By formula 3,

$$(p + q)(p - q) = p^2 - q^2$$

Observe that here no middle term appears.

4. Factor $x^2 + 2xy + y^2$.

Process.

$$x^2 + 2xy + y^2 = (x + y)^2 = (x + y)(x + y).$$

Suggestions.

1. Formula 1 applies.
2. The sign of the middle term, +, appears in the factors.

5. Factor $m^2 - 2mn + n^2$.

Process.

$$m^2 - 2mn + n^2 = (m - n)^2 = (m - n)(m - n).$$

Suggestions.

1. Formula 2 applies.
2. The sign of the middle term, -, appears in the factors.

6. Factor $p^2 - q^2$.

Process.

$$p^2 - q^2 = (p + q)(p - q).$$

Suggestions.

1. Formula 3 applies.
2. + appears in one factor, - in the other.

EXERCISES.

Square :

- | | |
|----------------|--------------------|
| 1. $b + c$. | 6. $4m - 3n$. |
| 2. $b - c$. | 7. $2a^2 + 3x^2$. |
| 3. $a + x$. | 8. $x - 5$. |
| 4. $a + 2b$. | 9. $3a + 5$. |
| 5. $3x - 2d$. | 10. $7 - 2x^2$. |

Expand :

11. $(b + d)(b - d)$.
12. $(4x - 3)(4x + 3)$.
13. $(2a^2 + 2b^2)(2a^2 - 2b^2)$.
14. $(4a - c)(4a + c)$.
15. $(a + 7b^2)(a - 7b^2)$.

Factor :

1. $x^2 - 9$.

Process.

$$x^2 - 9 = (x + 3)(x - 3).$$

Explanation.

1. $\sqrt{x^2} = x, \sqrt{9} = 3$.
2. $\therefore x^2 - 9 = (x + 3)(x - 3)$.

- | | |
|-----------------------------|--|
| 2. $a^2 + 6ab + 9b^2$. | 9. $36m^2 - 36mn + 9n^2$. |
| 3. $x^2 - 2x^2y + x^2y^2$. | 10. $x^6 + 8x^5 + 16x^4$. |
| 4. $1 - x^2$. | 11. $9x^2 - 16y^2$. |
| 5. $25 - 16a^2$. | 12. $25a^2 - b^4$. |
| 6. $4x^2 - 12ax + 9a^2$. | 13. $49m^2 - 100n^6$. |
| 7. $a^2 + 10ab + 25b^2$. | 14. $144a^2b^4 - 225c^6$. |
| 8. $x^2 - 16x + 64$. | 15. $4a^4b^2 + 52a^3b^3 + 169a^2b^4$. |

The above formula may be applied to certain compound expressions.

1. Factor $(a + b)^2 - c^2$.

Process.

Explanation.

$$(a + b)^2 - c^2 =$$

1. $\sqrt{(a + b)^2} = a + b$, and $\sqrt{c^2} = c$.

$$(a + b + c)(a + b - c)$$

2. $\therefore (a + b)^2 - c^2 = (a + b + c)$

$$(a + b - c).$$

2. Factor $(a + b)^2 - (a - c)^2$.

Process.

Explanation.

$$(a + b)^2 - (a - c)^2$$

1. $\sqrt{(a + b)^2} = a + b$, and $\sqrt{(a - c)^2} = a - c$.

$$(2a + b - c)(b + c)$$

2. $\therefore (a + b)^2 - (a - c)^2 = (a + b + a - c)(a + b - a + c) = (2a + b - c)(b + c).$

EXERCISES.

Factor :

- $(x - y)^2 - z^2$.
- $4y^2 - (2z - x)^2$.
- $(3a^2 - b^2)^2 - (a^2 - 3b^2)^2$.
- $(a^2 - 2bc)^2 - 4b^2c^2$.
- $m^2 - (x - y)^2$.
- $(a + b)^2 - (c + d)^2$.
- $(3a + 5)^2 - (2a - 3)^2$.
- $(x + 1)^2 - 1$.
- $(a - b)^2 - (c - d)^2$.
- $(7x + 4y)^2 - (2x + 3y)^2$.

3. Factor $a^2 + 2ab + b^2 - c^2$.

Suggestion.— $a^2 + 2ab + b^2 - c^2 = (a + b)^2 - c^2$.

4. Factor $a^2 - 2ab + b^2 - 5^2$.

Suggestion.— $a^2 - 2ab + b^2 - 5^2 = (a - b)^2 - 5^2$.

5. Factor $x^2 + 2xy + y^2 - z^2 - 2zn - n^2$.

Suggestion.— $(x^2 + 2xy + y^2) - (z^2 + 2zn + n^2) = (x + y)^2 - (z + n)^2$.

EXERCISES.

Factor :

1. $a^2 + 2a + 1 - d^2$.

2. $x^2 - 2x + 1 - q^2$.

3. $a^2 + b^2 - 2ab - c^2$.

4. $x^2 - 2bx - a^2 + b^2$.

5. $c^2 - 1 + d^2 + 2cd$.

6. $a^2 - 2am + m^2 - b^2 - 2bn - n^2$.

7. $a^2 - b^2 + m^2 - n^2 + 2am + 2bn$.

8. $a^2 - 2ad - b^2 - c^2 + d^2 + 2bc$.

9. $1 - 2x + x^2 - 4 + 4a - a^2$.

10. $a^2 + 2ax + x^2 - 4a^2b^4 - 8ab^4x - 4b^4x^2$.

Compound Expressions Containing a Common Monomial Factor.

1. Factor $abx^2 - aby^2$.

Explanation.

1. By inspection we find that ab is a common factor

2. Introducing a parenthesis, we have,

$$ab(x^2 - y^2) = ab(x + y)(x - y).$$

2. Factor $c^4 - c^2 + c^2d^2 + 2c^3d$.

Introducing a parenthesis, we have,

$$c^2(c^2 - 1 + d^2 + 2cd) = c^2[(c^2 + 2cd + d^2) - 1] = c^2[(c + d)^2 - 1] = c^2(c + d + 1)(c + d - 1).$$

EXERCISES.

Factor :

1. $x^3 + 2x^2y + xy^2 - 4x.$
2. $a^2y - 2aby + b^2y - c^2y.$
3. $4bm - m - 4b^2m + 4m^5.$
4. $5a^2 - 10am + 5m^2 - 5b^2 - 10bn - 5n^2.$
5. $q^2a^2 - q^2b^2 + q^2c^2 - q^2d^2 + 2acq^2 - 2bdq^2.$
6. $x^4y - x^2y^3 - x^3y^2 + xy^4.$

A Common Binomial Factor.

1. Factor
- $a^2 - y^2 - am - my.$

Process.

$$\begin{aligned} a^2 - y^2 - am - my &= \\ (a + y)(a - y) - m(a + y) &= \\ (a + y)(a - y - m). \end{aligned}$$

Explanation.

Separating the expression into two binomial parts, $a^2 - y^2$ and $am - my$, we find by inspection that $a + y$ is combined as a factor with both $a - y$ and m ; hence, we write $(a + y)$ as one factor and $a - y - m$ as the other factor.

2. Factor
- $a^2 + ab + ac + bc.$

Process.

$$\begin{aligned} a^2 + ab + ac + bc &= \\ a(a + b) + c(a + b) &= \\ (a + b)(a + c) \end{aligned}$$

Explanation.

Separating the expression into two binomial parts $a^2 + ab$ and $ac + bc$, we find by inspection that $a + b$ is combined as a factor with both a and c ; hence, we write $a + b$ as one factor, and $a + c$ as the other factor.

3. Factor
- $12a^2 - 4ab - 3ax^2 + bx^2.$

$$\begin{aligned} \text{Solution.} - 12a^2 - 4ab - 3ax^2 + bx^2 &= (12a^2 - 3ax^2) - (4ab - bx^2) = \\ 3a(4a - x^2) - b(4a - x^2) &= (4a - x^2)(3a - b). \end{aligned}$$

EXAMPLES.

1. $6x^2 + 3xy - 2ax - ay.$
2. $x^4 + x^3 + 2x + 2.$
3. $mx - 2my - nx + 2ny.$
4. $y^3 - y^2 + y - 1.$

5. $amx^2 + bmaxy - anxy - bny^2$.
6. $a^3x - ax^3 - a^2m + x^2m$.
7. $4x^3 - 2x^2 - 2ax + a$.
8. $y^2 - 100 - y - 10$.
9. $x + y - x^2 - 2xy - y^2$.
10. $c^2 - x^2 - (-c^2 - x^2 - 2xc)$.

A Special Case.

In a trinomial perfect square, as $x^2 + 2xy + y^2$, or $x^2 - 2xy + y^2$, the middle term $2xy$ is twice the product of $\sqrt{x^2}$ or x and $\sqrt{y^2}$, or y .

$x^4 + x^2y^2 + y^4$ is not, then, a perfect square, but by adding x^2y^2 to its middle term we have $x^4 + 2x^2y^2 + y^4$, a perfect square. Now, to preserve the original value, we subtract x^2y^2 and have $x^4 + 2x^2y^2 + y^4 - x^2y^2$, or $(x^2 + y^2)^2 - x^2y^2 = (x^2 + y^2 + xy)(x^2 + y^2 - xy)$.

EXERCISES.

Factor :

- | | |
|---------------------------------|-----------------------------------|
| 1. $x^4 + x^2 + 1$. | 6. $9x^4 - 4x^2b^2 + 4b^4$. |
| 2. $x^4 + x^2a^2 + a^4$. | 7. $25x^4 - 36x^2y^2 + 4y^4$. |
| 3. $4a^4 - 13a^2 + 1$. | 8. $49a^4 - 169a^2y^2 + 144y^4$. |
| 4. $9a^4 - 15a^2 + 1$. | 9. $225a^4 - 46a^2b^2 + 49b^4$. |
| 5. $49b^4 - 11b^2d^2 + 25d^4$. | 10. $256c^4 - 36c^2d^2 + 25d^4$. |

Factorable Trinomials that are not Perfect Squares.

Expanding the expression composed of the factors $(x + 2)(x + 3)$, we have $x^2 + 5x + 6$; from which we find by inspection that the elements of the trinomial and its factors are x , 2, and 3, combined in the trinomial as follows:

$x \cdot x = x^2$; $2 \times 3 = 6$; $(2 + 3)x = 5x$, the middle term, which alone contains each of the three elements.

1. Factor $x^2 + 7x + 12$.

Process.

Elements : $x, 3, 4$.

Factors : $(x + 3)(x + 4)$.

Proof:

$$\begin{array}{r} x + 3 \\ x + 4 \\ \hline x^2 + 3x \\ 4x + 12 \\ \hline x^2 + 7x + 12 \end{array}$$

Explanation.

1. $3x + 4x = 7x$, and $3 \times 4 = 12$.

2. Hence, the elements are $x, 3, 4$.

3. Writing the sum of x and 3 in a parenthesis as one factor, and the sum of x and 4 in a parenthesis as a second factor, we have for a result $(x + 3)(x + 4)$, the factors required.

2. Factor $x^2 - 8x + 15$.

Process.

Elements : $x, -3, -5$.

Factors : $(x - 3)(x - 5)$.

Proof:

$$\begin{array}{r} x - 3 \\ x - 5 \\ \hline x^2 - 3x \\ - 5x + 15 \\ \hline x^2 - 8x + 15 \end{array}$$

Explanation.

1. $-3x + (-5x) = -8x$; $-3x(-5) = 15$.

2. Hence, the elements are $x, -3, -5$.

3. Writing the sum of x and -3 and the sum of x and -5 in parenthesis as factors, we have for the required result $(x - 3)(x - 5)$.

3. Factor $x^2y^2 + 6xy - 55$.

Process.

Elements : $xy, -5, +11$.

Factors : $(xy - 5)(xy + 11)$.

Proof:

$$\begin{array}{r} xy - 5 \\ xy + 11 \\ \hline x^2y^2 - 5xy \\ + 11xy - 55 \\ \hline x^2y^2 + 6xy - 55 \end{array}$$

Explanation.

1. $11xy - 5xy = 6xy$; $-5 \times 11 = -55$.

2. Hence, the elements are $xy, -5, 11$.

3. Combining these elements in binomial factors, we have $(xy - 5)(xy + 11)$, the result required.

4. Factor $x^2 - 4xy - 45y^2$.

Process.

Elements : $x, 5y, -9y$.

Factors : $(x + 5y)(x - 9y)$.

Proof :

$$\begin{array}{r} x + 5y \\ x - 9y \\ \hline x^2 + 5xy \\ - 9xy - 45y^2 \\ \hline x^2 - 4xy - 45y^2 \end{array}$$

Explanation.

1. $-9xy + 5xy = -4xy$; $5y \times (-9y) = -45y^2$.

2. Hence, the elements are $x, 5y, -9y$.

3. Forming binomial factors, we have $(x + 5y)(x - 9y)$, the result required.

EXERCISES.

1. $x^2 + 6x + 9$.

6. $x^4 - 7x^2 - 18$.

2. $x^2 + 3x + 2$.

7. $x^4 + x^2y - 12y^2$.

3. $x^2 + 17x + 70$.

8. $a^6 - 12a^5 + 35a^4$.

4. $x^2 - x - 2$.

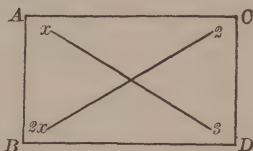
9. $x^2 - 30x + 200$.

5. $x^2 + 5ax + 6a^2$.

10. $3a^3b + 18a^2b + 27ab$.

Trinomials Less Easily Factored.

1. If the factors of a trinomial are $(x + 2)(2x + 3)$, the trinomial may be developed from its factors by arranging the terms of the factors about a rectangle as follows : x at A , 2 at C , $2x$ at B , and 3 at D .



1. Guided by the vertical lines we have $x \cdot 2x = 2x^2$, and $2 \cdot 3 = 6$, the first and third terms of the trinomial.

2. Guided by the diagonals, we have $x \cdot 3 + 2 \cdot 2x = 3x + 4x = 7x$, the middle term of the trinomial.

3. Hence, the trinomial is $2x^2 + 7x + 6$.

Having given the trinomial to determine its factors, we reverse the process.

$$2x^2 = x \cdot 2x ; 6 = 2 \cdot 3.$$

1. We place x at A , and $2x$ at B , on a rectangle as before.

2. By placing 2 at C , and 3 at D , the diagonals give $x \cdot 3 + 2x \cdot 2 = 3x + 4x = 7x$.

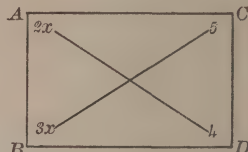
3. Hence, the horizontal lines give us $(x + 2)(2x + 3)$ the factors required.

2. Factor $6x^2 + 23x + 20$.

Explanation.

1. $6x^2 = 2x \cdot 3x$.

2. $20 = 4 \cdot 5$.



3. The diagonals give us $\left. \begin{array}{l} 2x \cdot 4 = 8x \\ \text{and } 3x \cdot 5 = 15x \end{array} \right\} = 23x$, the middle term.

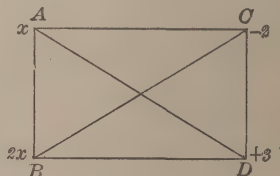
4. Hence, $6x^2 + 23x + 20 = (2x + 5)(3x + 4)$, the factors required.

3. Factor $2x^2 - x - 6$.

Explanation.

1. $2x^2 = x \cdot 2x$.

2. $-6 = -2 \cdot 3$.



3. The diagonals give us $\left. \begin{array}{l} x \cdot 3 = 3x \\ 2x \cdot -2 = -4x \end{array} \right\} = -x$, the middle term.

4. Hence, the factors are $(x - 2)(2x + 3)$.

Or, we may proceed as follows :

4. Factor $6x^2 + 19x + 10$.

Process.

$$6x^2 + 19x + 10 = (\quad) (\quad)$$

$$(2x + 5)(3x + 2)$$

Proof:

$$2x + 5$$

$$3x + 2$$

$$\hline 6x^2 + 15x$$

$$4x + 10$$

$$\hline 6x^2 + 19x + 10$$

Explanation.

1. We write two vacant parentheses ()

$$(\quad) (\quad)$$

2. $6x^2 = 2x \cdot 3x$. We write $2x$ in the first parenthesis and $3x$ in the other, and have $(2x \quad)(3x \quad)$.

3. $10 = 2 \cdot 5$. We write $+5$ in the first parenthesis and $+2$ in the other, and have

$$\overbrace{(2x + 5)(3x + 2)}, \text{ the factors required.}$$

4. For $2x \cdot 2 + 3x \cdot 5 = 4x + 15x = 19x$, the middle term.

EXERCISES.

Factor :

- | | |
|------------------------|-------------------------------------|
| 1. $12x^2 + 24x + 12.$ | 6. $5x^2 - 57x + 22.$ |
| 2. $6x^2 + 27x + 30.$ | 7. $-6x^2 + 7x + 20.$ |
| 3. $6x^2 - x - 12.$ | 8. $(a^2 - b^2) - (a + 5b) - 6.$ |
| 4. $4x^2 - 10x - 6.$ | 9. $(axy) + (acx + by) + bc.$ |
| 5. $15x^2 + 32x - 7.$ | 10. $a^4b^3 + (10a^4 - 9b^3) - 90.$ |

Additional Formulæ.

Odd Powers.

1. $\frac{a^3 + b^3}{a + b} = a^2 - ab + b^2$; hence, $a^3 + b^3 = (a + b)(a^2 - ab + b^2).$
2. $\frac{a^3 - b^3}{a - b} = a^2 + ab + b^2$; hence, $a^3 - b^3 = (a - b)(a^2 + ab + b^2).$
3. $\frac{a^5 + b^5}{a + b} = a^4 - a^3b + a^2b^2 - ab^3 + b^4$; hence, $a^5 + b^5 = (a + b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4).$

Even Powers.

4. $\left. \begin{matrix} a^2 + b^2 \\ a^4 + b^4 \\ a^8 + b^8 \end{matrix} \right\}$ are prime quantities and cannot be factored.
5. $a^4 - b^4 = (a^2 + b^2)(a^2 - b^2) = (a^2 + b^2)(a + b)(a - b).$
6. $a^6 - b^6 = (a^3 + b^3)(a^3 - b^3) = (a + b)(a^2 - ab + b^2)(a - b)(a^2 + ab + b^2).$
7. $a^6 + b^6 = (a^2 + b^2)(a^4 - a^2b^2 + b^4).$

Illustrations.

1. Factor $a^3 - m^3.$

Process.

$$a^3 - m^3 = (a - m)(a^2 + am + m^2)$$

Suggestions.

1. Formula 2 applies.
2. The terms of the second factor are all positive.

2. Factor $8a^3 - 1$.**Process.**

$$8a^3 - 1 = 2^3a^3 - 1 = \\ (2a - 1)(4a^2 + 2a + 1)$$

Suggestions.

1. Formula 2 applies.
2. $2a$ takes the place of a , and 1 the place of b .

3. Factor $27y^3 + 1$.**Process.**

$$27y^3 + 1 = 3^3y^3 + 1 = \\ (3y + 1)(9y^2 - 3y + 1)$$

Suggestions.

1. Formula 1 applies.
2. The second term of the second factor is negative.

4. Factor $a^{12} + m^{12}$.**Process.**

$$a^{12} + m^{12} = \\ (a^4 + m^4)(a^8 - a^4m^4 + m^8)$$

Suggestions.

1. Formula 7 applies.
2. $12 = 3 \times 4$; it is the even factor 4 that is used.

5. Factor $(a + 1)^3 - 8$.**Process.**

$$(a + 1)^3 - 8 = (a + 1)^3 - 2^3 = \\ [(a + 1) - 2][(a + 1)^2 + 2(a + 1) + 2^2] = \\ (a - 1)(a^2 + 4a + 7)$$

Suggestions.

1. Formula 2 applies.
2. $a + 1$ takes the place of a , and 2 the place of b .

3. Simplification gives the factors required.**EXERCISES.**

1. $m^3 + n^3$.

6. $x^6 - y^6$.

2. $a^3 + 64$.

7. $1 - x^8$.

3. $x^3y^3 - 64$.

8. $a^5 - b^5$.

4. $343 - x^3$.

9. $x^3 - y^3$.

5. $8a^3 - b^3$.

10. $(m + n)^3 + 8$.

Compound Expressions.**1. Factor $a^3 + y^3 + a + y$.****Process.**

$$(a^3 + y^3) + (a + y) = \\ (a + y)(a^2 - ay + y^2) + (a + y) = \\ (a + y)(a^2 - ay + y^2 + 1)$$

Suggestion.

The given expression is separable into two binomial parts which have the common factor $a + y$.

2. Factor $y^3 - b^3 - y + b$.

Process.

$$(y^3 - b^3) - (y - b) =$$

$$(y - b)(y^2 + yb + b^2) - (y - b) =$$

$$(y - b)(y^2 + yb + b^2 - 1)$$

Suggestion.

Here the common factor is $y - b$.

Factor :

EXERCISES.

1. $x^4 - y^4 + x^2 + y^2$.

2. $x^3 - a^3 + x - a$.

3. $8x^3 - y^3 - 2x + y$.

4. $a^3 + 8b^3 + a^2 + 5ab + 6b^2$.

5. $216 - a^3 + 36 - a^2$.

6. $m^4 - n^4 + 2m^3n - 2mn^3$.

7. $a^8 - b^8 - a^4 + b^4$.

8. $27a^3 - 27y^3 - (9a^2 - 18ay + 9y^2)$.

9. $x^3p^2 - 8y^3p^2 - 4x^3q^2 + 32y^3q^2$.

10. $a^9 - 64a^3 - a^6 + 64$.

Special Cases.

Polynomial Complete.

1. Factor $x^2 + 4y^2 - 4xy - 10x + 20y - 56$.

Explanation.

The first three terms constitute a perfect square; this square factored, the expression becomes $(x - 2y)(x - 2y) - 10x + 20y - 56$.

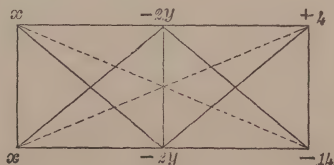
$$- 56 = - 14 \times 4.$$

The diagonals give $-10x$ and $+ 20y$.

Hence, we assume that the factors required are $(x - 2y + 4)(x - 2y - 14)$.

Process.

$$\begin{aligned} & \overbrace{x^2 - 4xy + 4y^2} - 10x + 20y - 56 = \\ & (x - 2y)(x - 2y) - 10x + 20y - 56 = \\ & (x - 2y + 4)(x - 2y - 14). \end{aligned}$$

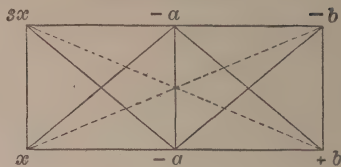


2. Factor $3x^2 - 4ax + a^2 + 2bx - b^2$.

Process.

$$\begin{aligned}
 &\overline{3x^2 - 4ax + a^2 + 2bx} \\
 &\quad - b^2 = \\
 &(3x - a)(x - a) + 2bx \\
 &\quad - b^2 = \\
 &(3x - a - b)(x - a + b).
 \end{aligned}$$

Explanation.

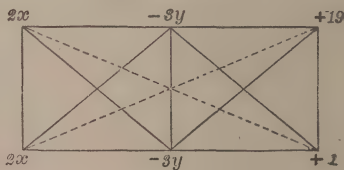


Explanation.

1. Factoring we have $3x^2 = 3x \cdot x$; $a^2 = -a \cdot -a$; $-b^2 = +b \cdot -b$.2. Having arranged these factors as in the diagram, the diagonals give us $-4ax$, 0, and $2bx$.3. Hence the horizontal lines exhibit the factors required: $3x - a - b$ and $x - a + b$.3. Factor $4x^2 + 9y^2 + 19 - 12xy + 40x - 60y$.

$$\begin{aligned}
 &\overline{4x^2 - 12xy + 9y^2 + 40x} \\
 &\quad - 60y + 19 = \\
 &(2x - 3y)(2x - 3y) \\
 &\quad + 40x - 60y + 19 = \\
 &(2x - 3y + 19)(2x - 3y \\
 &\quad + 1).
 \end{aligned}$$

Explanation.



EXERCISES.

Factor :

1. $4a^2 - 11ab + 17ac - 3b^2 + bc + 4c^2$.

2. $2x^2 + ax + cx - a^2 + 10ac - 21c^2$.

3. $3x^2 + 5xy - 7cx - 2y^2 + 7cy - 6c^2$.

4. $4a^2 + 10ab + 3ac - 6b^2 - 5bc - c^2$.

5. $8x^2 + 6xy + 10xz - 2y^2 + 2z^2$.

6. $2a^2 - 4ab - 4ac + 2b^2 + 4bc + 2c^2$.

7. $2a^2 + 6ax - 18a + 4x^2 - 30x + 36$.

8. $4a^2 - 11ab + 17ac - 3b^2 + bc + 4c^2$.

9. $3x^2 + 5xy - 7cx - 2y^2 + 7cy - 6c^2$.

Special Cases. Polynomial Incomplete.

1. Factor $x^3 + 3x^2 - 2$.

Process.

Explanation.

$$x^3 + 3x^2 - 2 = x^3 + 1 + 3x^2 - 3 =$$

$$(x^3 + 1) + 3(x^2 - 1) =$$

$$(x + 1)(x^2 - x + 1) + 3(x + 1)(x - 1) =$$

$$(x + 1)(x^2 - x + 1 + 3x - 3) =$$

$$(x + 1)(x^2 + 2x - 2).$$

We render the given expression factorable by adding 1 and subtracting 1. By adding 1, we have

$x^3 + 1 + 3x^2 - 2$. By subtracting 1, we have $x^3 + 1 + 3x^2 - 3$, an easily factorable form.

2. Factor $x^3 - 3x^2 + 4$.

Process.

Explanation.

$$x^3 - 3x^2 + 4 = x^3 + 1 - 3x^2 + 3 =$$

$$(x^3 + 1) - 3(x^2 - 1) =$$

$$(x + 1)(x^2 - x + 1) - 3(x + 1)(x - 1) =$$

$$(x + 1)(x^2 - x + 1 - 3x + 3) =$$

$$(x + 1)(x^2 - 4x + 4) =$$

$$(x + 1)(x - 2)^2.$$

We render the given expression factorable by adding 1 and subtracting 1.

3. Factor $x^3 - x^2 - 4$.

Process.

Explanation.

$$x^3 - x^2 - 4 = x^3 - 4x - x^2 + 4x - 4 =$$

$$x(x^2 - 4) - (x - 2)^2 =$$

$$x(x + 2)(x - 2) - (x - 2)^2 =$$

$$(x - 2)(x^2 + 2x - x + 2) =$$

$$(x - 2)(x^2 + x + 2).$$

We render the given expression factorable by adding $4x$ and subtracting $4x$.

Factor :

EXERCISES.

1. $x^3 - 3x - 2$.

6. $x^3 - 3x^2 - 10x + 24$.

2. $x^4 - 3x^2 - 2x$.

7. $x^3 - 8x^2 + 17x - 10$.

3. $a^3 + 3a^2 - 4$.

8. $6x^3 - 23x^2 + 16x - 3$.

4. $x^3 - 7x - 6$.

9. $4x^3 - 19x + 15$.

5. $3x^3 + 7x^2 - 4$.

10. $2x^3 + 6x^2 + 3x^2 + 4x - 15$.

MISCELLANEOUS EXERCISES.

Factor :

1. $6a^3b^2c$.
2. $ax - a$.
3. $3a^2 - 6ab$.
4. $3m^3 - 12m^2$.
5. $a^2 + 2ab + b^2$.
6. $9x^6 + 6x^3y + y^2$.
7. $15a^3x - 10a^3y + 5a^3z$.
8. $a^3 - 9a$.
9. $a^4 - 16b^4$.
10. $y^2 - 25x^2$.
11. $36x^2 - 25b^2$.
12. $a^3 - a^2y + ay^2$.
13. $a^2 - 2a - 8$.
14. $m^2 + 2mn + n^2 - 1$.
29. $x^2 - 7x - 60$.
30. $ax^2 + bx^2 - 5bx - 5ax + 6b + 6a$.
31. $5x^4 - 15x^3 - 90x^2$.
32. $14a^2x^3 - 35a^3x^2 + 14a^4x$.
33. $27 + 12x + x^2$.
34. $9a^6 - 10a^4b^2 + a^2b^4$.
35. $9x^4 + 21x^2y^2 + 25y^4$.
36. $(ax + by)^2 - (ax - by)^2$.
37. $a^2 + b^2 - 2ab - c^2$.
38. $80x^2y^5 - 5x^6y$.
39. $x^{16} + x^8 + 1$.
40. $25a^4b^4 + 70a^2b^2c^2 + 49c^4$.
41. $(a^2 - b^2 - c^2)^2 - 4b^2c^2$.
42. $x^3 - ax^2 - a^2x + a^3$.
43. $x^3 - 3x^2 + 3x - 2$.
15. $x^4 + x^2y^2 + y^4$.
16. $x^6 - 64$.
17. $x^3 + 8y^3z^3$.
18. $a^2 - b^2 + a - b$.
19. $x^4 - 2x^3 - 1 + 2x$.
20. $x^2 + 13xy - 30y^2$.
21. $27a^3 - 8b^3$.
22. $27x^3 + 64y^3$.
23. $x^4 - (x - 2)^2$.
24. $x^3 - 9x^2 + 11x + 21$.
25. $x^4y - x^2y^3 - x^3y^2 + xy^4$.
26. $a^2 - 18ab - 40b^2$.
27. $a^4 - 7a^2 + 6$.
28. $(x + y)^2 - x^2 + y^2$.

44. $a^3 + a^2c + abc + b^2c - b^3$.
45. $m^7 + n^7$.
46. $a^4b^6 - c^8$.
47. $1 - x - x^3 + x^4$.
48. $100x^2y^4 - 81z^2$.
49. $x^2 - 2bx - a^2 + b^2$.
50. $a^2 - b^2 + 2bc - c^2$.
51. $x^4 - 29x^2 + 100$.
52. $a^6 - 1$.
53. $a^3 + b^3 + a + b$.
54. $a^2 - 9b^2 + a + 3b$.
55. $x^5 - 23x^4 + 132x^3$.
56. $x^2 + 2xy - a^2 - 2ay$.
57. $a^6 - 64y^6$.
58. $abx^3 + x + ab + 1$.
59. $36a^4 - 21a^2 + 1$.
60. $x^2 + ax + bx - ab - 2a^2$.
61. $a^2 - b^2 + a - b$.
62. $2a^2 + ab - 3b^2 - 4bc - ac - c^2$.
63. $a^4 - a^2 - b^4 + b^2$.
64. $(c + d)^3 + (c - d)^3$.
65. $4(x - y)^3 - (x - y)$.
66. $x^4y - x^2y^3 - x^3y^2 + xy^4$.
67. $ax^2 + bx^2 - 5bx - 5ax + 6b + 6a$.
68. $4(xy - ab)^2 - (x^2 + y^2 - a^2 - b^2)^2$.
69. $a^3 + 3a^2b + 3ab^2 + b^3$.
70. $a^3 - 3a^2b + 3ab^2 - b^3$.
71. $a^3 + 3a^2 + 3a + 1$.
72. $a^3 - 3a^2 + 3a - 1$.
73. $x^3 + 6x^2 + 12x + 8$.
74. $a^6 - 3a^4b^2 + 3a^2b^4 - b^6$.
75. $x^2 + y^2 + z^2 + 2xy + 2xz + 2yz$.
76. $x^2 + y^2 + z^2 - 2xy + 2xz - 2yz$.
77. $x^8y^4 + 7x^4y^2 + 12$.
78. $a^3 + b^3 + a^2 - ab + b^2$.
79. $x^3 - y^3 - 3xy(x - y)$.
80. $4a^4 - 9a^2 + 6a - 1$.
81. $(m - n)^2 - 12(m - n) + 27$.
82. $(a + b)^4 + 4a^2(a + b)^2 + 4a^4$.

83. $(x^2 + 4x + 8)^2 + 3x(x^2 + 4x + 8) + 2x^2$.
84. $x^8 - 1$.
85. $15x^3 - 5x^2 + 33x - 11$.
86. $x^3 - 6x^2 + 11x - 6$.
87. $x^8 + x^6y^2 + x^2y + y^3$.
88. $a^2 - 9 + 9b^2 - 6ab$.
89. $8(x + y)^3 - (2x - y)^3$.
90. $(b + c)^2 - 6a(b + c) + 5a^2$.
91. $x^2 - 2bx - a^2 + b^2$.
92. $a^2x^5(a^3 - x) - a^5x^2(x^3 - a)$.
93. $(a + b)^2 - 1 - ab(a + b + 1)$.
94. $8x^3 - 6xy(2x + 3y) + 27y^3$.
95. $(x^2 + 4)^2 - 16x^2$.
96. $a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 + 2b^2c^2$.
97. $x^3 + 3x^2 + 3x + 1$.
98. $x^2 + (a - b)x - ab$.
99. $x^2 - (c - d)x - cd$.
100. $4a^2 - 11ab + 17ac - 3b^2 + bc + 4c^2$.

Highest Common Factor.

1. A Factor of a quantity is an exact divisor of that quantity.

$$a^2 - b^2 = (a + b)(a - b);$$

$$\text{hence, } \frac{a^2 - b^2}{a + b} = a - b, \text{ and } \frac{a^2 - b^2}{a - b} = a + b.$$

2. A factor that is not resolvable is called a Prime Factor.

a , $a + b$, $a^2 + b^2$, are prime factors of their product.

3. A Common Factor is a factor belonging to each of two or more quantities.

a^2 is a common factor of a^2b^3c and a^3cy .

4. The Highest Common Factor (H. C. F.) of quantities is the product of all their different common prime factors.

a^3c is the H. C. F. of a^4c , a^3c , a^5cxy , being the product of the common prime factors $aaac$. $a^2 - b^2$ is the H. C. F. of $a^4 - b^4$ and $a^2 - b^2$, being the product of their common prime factors $a + b$ and $a - b$.

$$a^4 - b^4 = (a^2 + b^2)(a + b)(a - b)$$

$$a^2 - b^2 = (a + b)(a - b).$$

Monomials.

1. Find the H. C. F. of $21a^2b$, $35ab^2c$, $49a^4b^2d^3$.

Explanation.

$$21a^2b = 3.7. a.a.b.$$

$$35ab^2c = 5.7. a.b.b.c.$$

$$49a^3b^2d = 7.7. a.a.a.b.b.d.$$

H. C. F. = $7. a.b = 7ab$, these being the only common prime factors.

EXERCISES.

Find the H. C. F. of :

1. $12a^3b^4c^4$, $9a^2b^3c$, $6a^5b^2c$.

2. $18x^2y^4z^3$, $15xy^3z$.

3. $42a^2x^3$, $60a^2x^2$.

4. $abcde$, a^2bef , $b^2cde^2f^2$.

5. $a^3b^4c^4$, a^2b^3 , a^5bc^2 .

6. $17ab^2c$, $34a^2bc$, $51abc^2$.

7. $60a^2x^2$, $45ax^3$, $90a^2x^5y$.

8. $21a^4m^2$, $18a^3m^3$, $15a^2m^4$.

9. $64x^{11}$, $8x^9$, $32x^7$, $4x^5$.

Polynomials Easily Factored.

Find the H. C. F. of $a^4b^2 - a^2b^4$ and $a^4b^3 + a^3b^4$.

Explanation.

$$a^4b^2 - a^2b^4 = a^2b^2(a^2 - b^2) = a^2b^2(a + b)(a - b).$$

$$a^4b^3 + a^3b^4 = a^3b^3(a + b).$$

H. C. F. = $a^2b^2(a + b)$, the product of all the common prime factors of the given numbers.

EXERCISES.

Find the H. C. F. of :

1. $5x^5y - 15x^2y^2$ and $25x^3y^3 - 45xy^5$.

2. $x^2 - y^2$ and $x^3 + y^3$.

3. $x^2 - 1$ and $x^3 - 1$.

4. $x^3 + 1$ and $x^3 + mx^2 + mx + 1$.

5. $x^2 + x$, $x^2 - 1$, $x^2 - x - 2$.

6. $a^3b - ab^3$, $a^5b^2 - a^2b^5$.

7. $a^2 - 5ab + 4b^2$, $a^3 - 5a^2b + 4b^3$.

Suggestion.— $-5a^2b = -a^2b - 4a^2b$.

8. $a^2 + 2a + 1$, $a^3 + 2a^2 + 2a + 1$.

9. $x^3 - 1$, $x^2 - 1$, $(x - 1)^2$.

10. $10x^3y + 9x^2y^2 - 9xy^3$, $4xy^2 + 15y^3 - 4x^2y$.

11. $x^2 - 2xy + y^2$, $(x - y)^3$.

12. $x^3 + 8y^3$, $x^2 + xy - 2y^2$.

13. $15x^3 - 19x^2y + 6xy^2$, $10x^4 - x^3y - 3x^2y^2$.

14. $x^6 - y^6$, $x^4 + xy^3$, $x^6 + 2x^3y^3 + y^6$.

15. $20x^4 + x^2 - 1$, $75x^4 + 15x^3 - 3x - 3$.

Polynomials Less Easily Factored.

Find the H. C. F. of 128 and 144.

Process.

Explanation.

128)144(1

Treating 128 and 144 as divisor and dividend,

128

we have $144 = 1 \times 128 + 16$. Hence,

16)128(8

$\frac{144}{\text{H. C. F.}} = 1 \times \frac{128}{\text{H. C. F.}} + \frac{16}{\text{H. C. F.}}$. That is, the

128

H. C. F. of 144 and 128 = the H. C. F. of 128

0

and 16. $128 \div 16 = 8 + 0$; hence, 16 is the H.

C. F. required, for 16 is the highest number that is contained in both itself and 128.

In general, the dividend = the quotient $\times$ the divisor + the remainder ; that is, $D = Q \times d + R$.

Let it be required to find the H. C. F. of D and d .

Explanation.

$\frac{D}{H. C. F.} = Q \times \frac{d}{H. C. F.} + \frac{R}{H. C. F.}$; that is, the H. C. F. of D and d . = the H. C. F. of d and R .

When $\frac{d}{R} = Q' + 0$, R = the H. C. F. required; for R is the highest quantity contained in both itself and d .

When Q' does not equal 0, the H. C. F. required is, as before, the H. C. F. of the last remainder and the preceding divisor; and so we proceed till we find the remainder 0.

Find the H. C. F. of $a^2 - 5ab + 4b^2$ and $a^3 - 5a^2b + 4b^3$.

Explanation.

$$\begin{array}{r} d \qquad \qquad D \qquad \qquad Q \\ a^2 - 5ab + 4b^2 \overline{) a^3 - 5a^2b + 4b^3} (a \\ \underline{a^3 - 5a^2b + 4ab^2} \\ -4ab^2 + 4b^3. \end{array}$$

Since $-4b^2$ does not appear in the terms of the divisor, we reject it and have $R = a - b$.

$$\begin{array}{r} R \qquad \qquad d \qquad \qquad Q' \\ a - b \overline{) a^2 - 5ab + 4b^2} (a + 1 \\ \underline{a^2 - ab} \\ -4ab + 4b^2 \end{array}$$

Rejecting $-4b$, we have $a - b$; dividing again we have

$$R = \frac{a - b}{0}$$

Hence, $a - b$ is the H. C. F. required.

EXERCISES.

Find the H. C. F. of:

1. $6x^2 - x - 12$ and $6x^3 - 13x^2 - 6x + 18$.
2. $x^4 + a^3x - 2a^4$ and $x^4 - 2ax^3 + a^4$.
3. $a^4 - 4a^2 + 3a$ and $a^3 + 4a^2 - 5$.

Suggestion.—Reject a from the first polynomial.

4. $x^3 + 2x^2 + 2x + 1$ and $x^3 - 2x - 1$.

5. $x^4 + 2x^2 + 9$ and $7x^3 - 11x^2 + 15x + 9$.

Suggestion.—Multiply the first polynomial by 7; for the introduction of a factor not common cannot affect the H. C. F., which is the product of common factors only.

6. $24x^4 - 2x^3 - 60x^2 - 32x$ and $18x^4 - 6x^3 - 39x^2 - 18x$.

NOTE.— $2x$, rejected from the first polynomial, and $3x$, from the second, have a common factor x , which must be retained as a factor of the H. C. F. sought.

7. $x^3 + 2x^2 - 13x + 10$ and $x^3 + x^2 - 10x + 8$.

8. $x^3 - 5x^2 - 99x + 40$ and $x^3 - 6x^2 - 86x + 35$.

9. $6x^3 - 6x^2y + 2xy^2 - 2y^3$ and $12x^2 - 15xy + 3y^2$.

10. $4x^4 + 9x^3 + 2x^2 - 2x - 4$ and $3x^3 + 5x^2 - x + 2$.

11. $10x^3 + 35x^2 + 30x$ and $4x^4 + 26x^3 + 58x^2 + 42x$.

12. $2x^2 - 14x + 20$ and $20x - 25x^2 + 4x^3 + 25$.

13. $x^4 - 10x^3 + 35x^2 - 50x + 24$ and $x^3 - 7x^2 + 11x - 5$.

14. $x^3 + 2x^2 + 2x + 1$ and $x^3 - 4x^2 - 4x - 5$.

15. $8x^2 + 2x - 3$ and $6x^3 + 5x^2 - 2$.

16. $2x^3 - 3x^2 - 16x + 24$ and $4x^5 + 2x^4 - 28x^3 - 16x^2 - 32x$.

17. $64x^{11} - 8x^2y^6$ and $12x^8 + 3x^2y^4 - 12x^5y^2$.

18. $3a^4 - 2a^3b + 2a^2b^2 - 5ab^3 - 2b^4$ and $6a^4 - a^3b + 2a^2b^2 - 2ab^3 - b^4$.

When it is required to find the H. F. C. of more than two quantities, first find the H. C. F. of two of them, then of that H. C. F. and one of the remaining quantities, and so on for all that remain. The last H. C. F. will be the H. C. F. of all the quantities.

EXERCISES.

1. $x^3 - 9x^2 + 26x - 24$, $x^3 - 10x^2 + 31x - 30$,
 $x^3 - 11x^2 + 38x - 40$.

2. $2x^2 - 5x - 42, 4x^2 + 8x - 21, 6x^2 + 23x + 7.$

3. $x^3 - x^2 - 2x, x^3 - 2x^2 + 3x - 6, 2x^3 - 3x^2 - x - 2.$

4. $x^4 - 10x^2 + 9, x^4 + 10x^3 + 20x^2 - 10x - 21,$
 $x^4 + 4x^3 - 22x^2 - 4x + 21.$

5. $3x^3 - 7x^2y + 5xy^2 - y^3, x^2y + 3xy^2 - 3x^3 - y^3,$
 $3x^3 + 5x^2y + xy^2 - y^3.$

6. $x^2 - 3xy - 10y^2, x^2 + 2xy - 35y^2, x^2 - 8xy + 15y^2.$

Lowest Common Dividend.

1. A Dividend of a quantity contains all the prime factors of that quantity.

2. A Common Dividend of two or more quantities contains all the prime factors of each of those quantities.

3. A Lowest Common Dividend (L. C. Dd.) of two or more quantities contains all the different prime factors of those quantities, a common prime factor occurring but once.

4. What are all the different prime factors and the L. C. Dd. of 40, 56, and 60?

Process.

Explanation.

$$40 = 2 \cdot 2 \cdot 2 \cdot 5$$

The different prime factors

$$56 = 2 \cdot 2 \cdot 2 \cdot 7$$

are:

$$60 = 2 \cdot 2 \cdot 3 \cdot 5$$

2, common to all the given numbers.

$$\text{L. C. Dd.} = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5 \cdot 7 =$$

2, common to all the given numbers.

$$840.$$

2, common to some of the given numbers.

5, common to some of the given numbers.

3, not common.

7, not common.

Hence, the L. C. Dd. required is $2 \cdot 2 \cdot 2 \cdot 3 \cdot 5 \cdot 7 = 840.$

2. What are the different prime factors and the L. C. Dd. of $6a^3$ and a^2b^2 ?

Process.	Explanation.
$6a^3 = 2 \cdot 3 \cdot a \cdot a \cdot a$	The common factors are $a \cdot a$.
$a^2b^2 = a \cdot a \cdot b \cdot b$	The factors not common are
L. C. Dd. $= 2 \cdot 3 \cdot a \cdot a \cdot a \cdot b \cdot b$	$2, 3, a, b, b$.
$= 6a^3b^2.$	Hence the L. C. Dd. =
	$2 \cdot 3 \cdot a \cdot a \cdot a \cdot b \cdot b = 6a^3b^2.$

Monomials.

1. Find the L. C. Dd. of 24, 36, 60.

Process.	Explanation.								
Common factors. <table style="border-collapse: collapse;"> <tr> <td style="border-right: 1px solid black; padding: 5px;">2</td><td style="padding: 5px;">24, 36, 60</td></tr> <tr> <td style="border-right: 1px solid black; padding: 5px;">2</td><td style="padding: 5px;">12, 18, 30</td></tr> <tr> <td style="border-right: 1px solid black; padding: 5px;">3</td><td style="padding: 5px;">6, 9, 15</td></tr> <tr> <td style="border-right: 1px solid black; padding: 5px;"></td><td style="padding: 5px;">2, 3, 5</td></tr> </table>	2	24, 36, 60	2	12, 18, 30	3	6, 9, 15		2, 3, 5	By division we find that the factors common to all the numbers are 2, 2, 3, and the factors not common are 2, 3, 5. Hence the L. C. Dd. = $2 \cdot 2 \cdot 3 \cdot 2 \cdot 3 \cdot 5 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 = 360$.
2	24, 36, 60								
2	12, 18, 30								
3	6, 9, 15								
	2, 3, 5								

Factors not common.

$$\text{L. C. Dd.} = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 = 360.$$

2. Find the L. C. Dd. of $4a^3b$, $2ab^2$, $6ax$.

Process.	Explanation.						
Common factors. <table style="border-collapse: collapse;"> <tr> <td style="border-right: 1px solid black; padding: 5px;">$2a$</td><td style="padding: 5px;">$4a^3b, 2ab^2, 6ax$</td></tr> <tr> <td style="border-right: 1px solid black; padding: 5px;">b</td><td style="padding: 5px;">$2a^2b, b^2, 3x$</td></tr> <tr> <td style="border-right: 1px solid black; padding: 5px;"></td><td style="padding: 5px;">$2a^2, b, 3x$</td></tr> </table>	$2a$	$4a^3b, 2ab^2, 6ax$	b	$2a^2b, b^2, 3x$		$2a^2, b, 3x$	By division we find that 2, a and b are the common factors, and $2a^2$, b , $3x$ the factors not common. Hence the L. C. Dd. = $2ab \cdot 6a^2bx = 12a^3b^2x$.
$2a$	$4a^3b, 2ab^2, 6ax$						
b	$2a^2b, b^2, 3x$						
	$2a^2, b, 3x$						

Factors not common.

$$\text{L. C. Dd.} = 2ab \cdot 6a^2bx = 12a^3b^2x.$$

EXERCISES.

Find the L. C. Dd. of:

1. $6ab$, $10bc$, $14ac$.
2. $8a^2b^2$, $24a^4b^2c^2$, $18abc^3$.
3. $3a^2b$, $4ac^2$, $6b^2c$.
4. axy , ax^2 , a^2y^3 .

5. $2x^2y^3, 3xy, 4x^3y^4$.
6. $7a^2b, 4ac^2, 6ac, 21bc$.
7. $19x^3y^2, 57a^2b^2c^3, 171ax^3$.
8. $66a^4b^2c^3, 44a^3b^4c^2, 24a^2b^3c^4$.
9. $8a^2x^3, 36a^3x^3, 4a^2x^4, 10ax$.
10. $3x^3, 8, 6x^2y, 12xy^2, 21, 7xy$.

Polynomials Easily Factored.

Find the L. C. Dd. of $a^3 - y^3$ and $a^2 - 2ay + y^2$.

Explanation.

The common factor is $a - y$; the factors not common are $(a - y)$ and $(a^2 + ay + y^2)$.

Process.

$$a^3 - y^3 = (a - y)(a^2 + ay + y^2)$$

$$a^2 - 2ay + y^2 = (a - y)(a - y)$$

$$\begin{aligned} \text{L. C. Dd.} &= (a - y)(a - y)(a^2 + ay + y^2) \\ &= (a - y)^2(a^2 + ay + y^2). \end{aligned}$$

Hence the L. C. Dd. is $(a - y)(a - y)(a^2 + ay + y^2) = (a - y)^2(a^2 + ay + y^2)$.

EXERCISES.

Find the L. C. Dd. of:

1. $x^2, x^2 - 3x$.
2. $x^2 - 1, x^2 + x$.
3. $6x^2 - 2x, 9x^2 - 3x$.
4. $x^3 - x, x^3 - 1$.
5. $x^2 - 1, x^3 + 1$.
6. $x^3 - 3x + 2, x^3 + 2x^2 - x - 2$.
7. $6ab^2(a + b)^2$ and $4a^2b(a^2 - b^2)$.
8. $2axy(x - y)^2, 3ax^2(x^2 - y^2), 4y^2(x + y)^2$.
9. $a^3 - x^3, a^2 - x^2, a^2 + ax + x^2$.
10. $a^2 - ax + x^2, a^2 + ax + x^2, a^3 + x^3$.
11. $x - x^2, (1 - x)^2, x(1 - x)^3$.

12. $x^2 - y^2, x^2 - 3xy + 2y^2$.
 13. $a^2 - b^2, a^3 - b^3, a^3 + b^3$.
 14. $x^2 + x - 2, x^2 - 2x + 1$.
 15. $x^2 - y^2 - z^2 + 2yz, x^2 - y^2 + z^2 + 2xz$.
 16. $a^2 - x^2 - a - x, 5a^2b - 10abx + 5bx^2, a^2b - abx - ab$.
 17. $6x^2 - x - 1, 2x^2 + 3x - 2$.
 18. $2x^2 + xy - 6y^2, 3x^2 - 8xy + 4y^2$.

Polynomials Less Easily Factored.

1. Find the L. C. Dd. of 143 and 442.

Process.

$$\begin{array}{r}
 1. \quad 143 \overline{) 442} 3 \\
 \quad \quad 429 \\
 \hline
 \text{H. C. F.} = 13 \overline{) 143} 11 \\
 \quad \quad \quad 143 \\
 \quad \quad \quad \hline
 \quad \quad \quad 0
 \end{array}$$

$$2. \text{ H. C. F. } 13 \overline{) 143 \cdot 442}.$$

$$11 \cdot 34.$$

Factors not common.

$$3. \text{ L. C. Dd. } = 13 \times 11 \times 34 = 4862$$

Explanation.

1. We find the H. C. F. of the given numbers, and thus obtain the common prime factor 13.

2. Dividing the given numbers by 13, we have for quotients the factors not common, 11 and 34.

3. Hence, $13 \times 11 \times 34 = 4862$, is the L. C. Dd. required.

2. Find the L. C. Dd. of $2x^2 - xy - 6y^2$ and $3x^2 - 8xy + 4y^2$.

Process.

$$1. \text{ H. C. F. } = x - 2y.$$

$$\begin{array}{r}
 2. \quad x - 2y \overline{) 2x^2 - xy - 6y^2} \quad 3x^2 - 8xy + 4y^2 \\
 \quad \quad \quad 2x + 3y, \quad \quad \quad 3x - 2y. \\
 \hline
 \end{array}$$

Factors not common.

$$3. \text{ L. C. Dd. } = (x - 2y)(2x + 3y)(3x - 2y).$$

Explanation.

1. The H. C. F. of the two polynomials is $x - 2y$.
2. Dividing the two polynomials by $x - 2y$, we have for quotients $2x + 3y$ and $3x - 2y$, the factors not common.
3. Hence, the L. C. Dd. required = $(x - 2y)(2x + 3y)(3x - 2y)$.

RULE.

1. Find the G. C. F. of the two given polynomials.
2. Divide the polynomials by the G. C. F.
3. Find the product of the G. C. F. and the quotients.

EXERCISES.

Find the L. C. Dd. of :

1. $6x^2 + 13x - 28$ and $12x^2 - 31x + 20$.
2. $6x^2 - x - 1$ and $2x^2 + 3x - 2$.
3. $2x^2 + x - 6$ and $4x^2 - 8x + 3$.
4. $8x^2 + 30x + 7$ and $12x^2 - 29x - 8$.
5. $21x^2 - 26x + 8$ and $7x^3 - 4x^2 - 21x + 12$.
6. $x^3 + 5x^2 + 7x + 2$ and $x^2 + 6x + 8$.
7. $x^3 - 6x^2 + 11x - 6$ and $x^3 - 9x^2 + 26x - 24$.
8. $x^3 - 7x - 6$ and $x^3 + 8x^2 + 17x + 10$.
9. $2x^4 + x^3 - 20x^2 - 7x + 24$ and $2x^4 + 3x^3 - 13x^2 - 7x + 15$.
10. $x^2 - 5x + 6$, $x^3 - 3x - 2$.
11. $6x^3 + x^2 - 5x - 2$, $6x^3 + 5x^2 - 3x - 2$.
12. $x^4 + ax^3 + a^3x + a^4$, $x^4 + a^2x^2 + a^4$.
13. $x^3 - 5x^2 + 9x - 9$, $x^4 - 4x^2 + 12x - 9$.
14. $3x^3 - 13x^2 + 23x - 21$, $6x^3 + x^2 - 44x + 21$.
15. $x^4 - x^3 + 8x - 8$, $x^4 + 4x^3 - 8x^2 + 24x$.

When the L. C. Dd. of more than two quantities is required, first find the L. C. Dd. of two of them, and then the L. C. Dd. of the result and a third quantity, and so on.

1. $x^3 - 6x^2 + 11x - 6$, $x^3 - 9x^2 + 26x - 24$, $x^3 - 8x^2 + 19x - 12$.
2. $3a^2 - 2a - 1$, $6a^2 - a - 1$, $9a^2 - 3a - 2$.
3. $21x^3 + 28x^2 - 15x - 20$, $21x^3 - 28x^2 - 15x + 20$, $14x^3 - 21x^2 - 10x + 15$.
4. $4x^3 - 3x - 1$, $2x^3 - 3x^2 + 1$, $4x^3 - 3x^2 - 1$.
5. $x^3 - 5x^2 + 9x - 9$, $x^3 - x^2 - 9x + 9$, $x^4 - 4x^2 - 12x - 9$.

FRACTIONS.

1. A fraction is an indicated division, made by writing the dividend above a vinculum (dividing line), and the divisor below the vinculum.

2. The dividend is called the *Numerator*; the divisor is called the *Denominator*.

3. The numerator and denominator are called the *Terms* of the fraction.

4. A quantity having no fractional part is called an *Integer*, or *Integral Quantity*.

5. A quantity having both an integral and a fractional part is called a *Mixed Quantity*; as $a + \frac{x}{y}$.

6. The *Transformation* (Reduction) of a fraction changes its form without changing its value.

A fraction is to be regarded as affected by three signs, namely, by $+$ or $-$ standing before the numerator, the denominator, and the vinculum; thus, $\pm \frac{\pm a}{\pm b}$.

7. A fraction may be transformed:

1. By change of signs; 2. By change to *higher or lower terms*; 3. By change to a whole or a mixed number.

The Law of Signs.

We may have the following transformations :

$$1. \quad +\frac{+a}{+b} = +\frac{-a}{-b} = -\frac{+a}{-b} = -\frac{-a}{+b} = m, \text{ and} \\ -\frac{+a}{-b} = -\frac{+a}{+b} = +\frac{+a}{-b} = +\frac{-a}{+b} = -m.$$

That is, of the three signs belonging severally to the numerator, the denominator, and the vinculum, *any two may be changed* without affecting the value or quality of the fractional expression. Hence, if only the sign of the numerator or of the denominator be changed, the sign of the vinculum must be changed.

$$2. \quad \frac{ab}{cd} = \frac{(-a)(-b)}{c \quad d} = \frac{(-a)(-b)}{(-c)(-d)} = +m, \text{ and} \\ \frac{(-a)b}{cd} = \frac{(-a)(-b)}{(-c) \quad d} = -m.$$

That is, of the signs belonging to the factors of the terms of a fraction, an *even* number may be changed without affecting the quality or value of the fractional expression. Hence, if an *odd* number of factor signs be changed, the sign of the vinculum must be changed.

EXERCISES.

1. Assuming $\frac{a}{b} = m$, transform the fraction variously, letting $+m$ remain the quotient.

2. Assuming $\frac{x}{y} = +v$, transform the fraction variously, letting $-v$ become the quotient.

3. Assuming $\frac{m}{n} = d$, correct the following erroneous equations :

$$\begin{aligned}
 + \frac{-m}{-n} &= -d; & + \frac{+m}{-n} &= +d; & - \frac{-m}{+n} &= -d; \\
 + \frac{+m}{-n} &= +d; & - \frac{-m}{-n} &= +d; & + \frac{+m}{+n} &= -d.
 \end{aligned}$$

4. Why is $-\frac{b-a}{c}$ identical with $\frac{a-b}{c}$?

NOTE.—When a change of sign is made, every sign of a polynomial numerator or denominator must be changed.

5. Why is $-\frac{c-a+b}{d} = \frac{a-c-b}{d}$ a correct equation?

6. Explain why $a + \frac{a-d}{e}$ has the same value as $a - \frac{d-a}{e}$.

7. By changing signs, make the denominators *like* in each expression.

$$\frac{a}{b-a} + \frac{b}{a-b} - \frac{c}{-a+b}; \quad \frac{1}{a(a-b)(a-c)} + \frac{1}{a(b-a)(c-a)}.$$

8. Show that $\frac{a-x}{(a-b)(c-b)(c-a)} = \frac{a-x}{(a-b)(b-c)(a-c)}$ is an *Identical Equation*, i. e., that the first member may assume the exact form of the second member.

9. Transform $\frac{y-x}{(b-a)(x-y)(a-b)}$, so that the numerator and denominator may have a common factor, and the denominator may have $(a-b)^2$.

10. Transform $-\frac{a-b}{(a+b)(c-b)(b-a)}$, so that $c-b$ shall become $b-c$, and $a-b$ a common factor.

The Law of Higher and Lower Terms.

1. To prove that $\frac{a}{b} = \frac{am}{bm}$.

1. If $\frac{a}{b} = x$.

2. Then $a = bx$, for $D = D \times Q$.

3. $am = bmx$ Ax. 3.

4. $\frac{am}{bm} = x$. Ax. 4.

5. $\therefore \frac{a}{b} = \frac{am}{bm}$. The values of x in 1 and 4.

Hence we have the principle, *Multiplying or dividing both terms of a fraction by the same quantity does not alter the value of the fraction.*

On this principle, the terms of a fraction are made higher by the *introduction* of a common factor, or of common factors, and are made lower by the *removal* of a common factor, or of common factors.

Obviously, the terms of a fraction are made *lowest* by the removal of *all* their common factors.

The Terms Easily Factored.

Reduction to Lowest Terms.

RULE.

Resolve the terms of the fraction into their prime factors, and cancel all the common factors.

1. Reduce to its Lowest Terms $\frac{42m^2b}{49mn^2}$.

Process.

$$\frac{42m^2b}{49mn^2} = \frac{2 \cdot 3 \cdot 7m^2b}{7 \cdot 7mn^2} = \frac{6mb}{7n^2}, \text{ Ans.}$$

2. Reduce to its Lowest Terms $\frac{x^3 - 27}{x^2 + 2x - 15}$.

Process.

$$\frac{x^3 - 27}{x^2 + 2x - 15} = \frac{(x - 3)(x^2 + 3x + 9)}{(x + 5)(x - 3)} = \frac{x^2 + 3x + 9}{x + 5}, \text{ Ans.}$$

3. Reduce $\frac{(x-1)(1-y)}{(1-x)(y-1)(7+a)}$ to its Lowest Terms.

$$\frac{(x-1)(1-y)}{(1-x)(y-1)(7+a)} = \frac{(x-1)(y-1)}{(x-1)(y-1)(7+a)} = \frac{1}{7+a}, \text{ Ans.}$$

Observe that the numerator is 1, not 0.

EXERCISES.

Reduce to Lowest Terms :

$$1. \frac{58ab^2c^2}{87a^4b^3c^2}.$$

$$6. \frac{20(x^3 - y^3)}{5x^2 + 5xy + 5y^2}.$$

$$2. \frac{12x^5y^2z^3}{30xy^3z^3}.$$

$$7. \frac{(xy - 3y^2)^2}{x^3y^2 - 27y^5}.$$

$$3. \frac{3a^2x}{6a^2 - 9a^3x}.$$

$$8. \frac{5a^3b + 10a^2b^2}{3a^2b^2 + 6ab^3}.$$

$$4. \frac{2a^2cd + 2abcd}{6a^2xy + 6abxy}.$$

$$9. \frac{x^2 - ax}{a^2 - x^2}.$$

$$5. \frac{ax}{a^2x^2 - ax}.$$

$$10. \frac{5x^2 + 4x - 1}{5x^2 + 19x - 4}.$$

The Terms Less Easily Factored.

Reduce to Lowest Terms :

$$1. \frac{x^4 - 9x^2}{x^4 - x^3 - 6x^2}.$$

$$3. \frac{a^4 - 14a^2 - 51}{a^4 - 2a^2 - 15}.$$

$$2. \frac{10a^2 - a - 21}{2a^2 - 7a + 6}.$$

$$4. \frac{x^3 - 6x - 4}{3x^3 - 8x + 8}.$$

$$5. \frac{2x^2 + 17x + 21}{3x^2 + 26x + 35}.$$

$$6. \frac{3x^2 + 23x + 14}{3x^2 + 41x + 26}.$$

$$7. \frac{a^2 - (b - c)^2}{(a + c)^2 - b^2}.$$

$$8. \frac{x^3 - x^2 + 2}{x^3 - 3x^2 + 4x - 2}.$$

$$9. \frac{2x^3 + 3x^2 - 1}{x^4 + 2x^3 + 2x^2 + 2x + 1}.$$

$$10. \frac{x^{3m} + x^{2m} - 2}{x^{2m} + x^m - 2}.$$

Reduction to an Entire or Mixed Quantity.

A fraction being an indicated division, the entire or mixed quantity is simply the resulting quotient.

Hence the rule: Divide the numerator by the denominator.

Observe the following:

$$1. \frac{bx - cx + m}{x} = b - c + \frac{m}{x}.$$

$$2. \frac{bx - cx - m}{x} = b - c - \frac{m}{x}. \quad \text{The fractional part may}$$

assume four different forms: $-\frac{m}{x}, +\frac{-m}{x}, +\frac{m}{-x}, -\frac{-m}{-x}.$

$$3. \frac{bx + c - m}{x} = b + \frac{c - m}{x}.$$

$$4. \frac{bx - c - m}{x} = b - \frac{c + m}{x}. \quad \text{Why have } c \text{ and } m \text{ in the}$$

quotient suffered change of sign?

Write all possible transformations of the fractional part.

Can a parenthesis be employed? Which form is preferable?

EXERCISES.

Reduce to an entire or mixed quantity:

$$1. \frac{x^2 - 3x + 4}{x}$$

$$3. \frac{x^2 + y^2}{x + y}$$

$$2. \frac{6x^2 - 12x + 5}{6x}$$

$$4. \frac{x^3 + x^2 - 1}{x^2}$$

$$5. \frac{x^2 + 6x + 12y - 5y^2}{x - 2y + 6}.$$

$$8. \frac{2x^2 - 6x - 1}{x - 3}.$$

$$6. \frac{x^2 + x}{x - 1}.$$

$$9. \frac{x^5 - x^3 - 2x^2 - 2x - 1}{x^2 - x - 1}.$$

$$7. \frac{4ax - 2x^2 - a^2}{2a - x}.$$

$$10. \frac{x^4 + 1}{x - 1}.$$

Reduction of a Mixed Quantity to the Fractional Form.

This case exactly reverses the preceding one; hence the

RULE:

1. Multiply the integral part by the denominator.
2. To the product add the numerator when the sign of the fraction is +; from the product subtract the numerator when the sign is -.
3. Under the result write the denominator.

$$1. \text{ Reduce to fractional form } a + b + \frac{a^2 + b^2}{a - b}.$$

Process.

$$a + b + \frac{a^2 + b^2}{a - b} = \frac{a^2 - b^2 + a^2 + b^2}{a - b} = \frac{2a^2}{a - b}.$$

$$2. \text{ Reduce to fractional form } 2 + 3y - \frac{y - 5}{4y}.$$

Process.

$$2 + 3y - \frac{y - 5}{4y} = \frac{8y + 12y^2 - y + 5}{4y} = \frac{12y^2 + 7y + 5}{4y}.$$

EXERCISES.

Reduce to fractional form :

$$1. y + \frac{a^2 - y^2 - c}{y}.$$

$$6. 1 - \frac{x^2 + y^2}{x^2 - y^2}.$$

$$2. y - \frac{a^2 - y^2 - c}{y}.$$

$$7. x - 1 - \frac{x - 2}{x^2 + 2x - 1}.$$

$$3. x + 1 + \frac{1}{x - 1}.$$

$$8. a + b - \frac{a^2 + b^2}{a - b}.$$

$$4. x - 1 - \frac{2}{x - 2}.$$

$$9. 1 + x + x^2 + x^3 + \frac{x^4}{1 - x}.$$

$$5. \frac{3x - 2}{x + 1} - x - 2.$$

$$10. \frac{x^4}{x^2 + x - 1} - x^2 + x - 1.$$

Reduction of Fractions to a Lowest Common Denominator.

1. A common denominator of two or more fractions is a common dividend of their denominators.

2. The lowest common denominator (L. C. D.) of two or more fractions is the lowest common dividend (L. C. Dd.) of their denominators.

3. Reduction to the L. C. D. depends upon the principle already stated : multiplying both terms of a fraction by the same quantity does not change the value of the fraction.

RULE.

1. Find the L. C. Dd. of the denominators.
2. Divide this by the denominators severally.
3. Multiply both terms of each fraction by the quotient obtained by its denominator.

$$1. \text{ Reduce to L. C. D. } \frac{a - b}{ab}, \frac{c - a}{ac}, \frac{b - c}{bc}.$$

1. The L. C. Dd. of the denominators = abc .

2. $abc \div ab, ac, bc = c, b, a$.

$$3. \frac{(a-b)c}{ab \times c}, \frac{(c-a)b}{ac \times b}, \frac{(b-c)a}{bc \times a} = \frac{ac-bc}{abc}, \frac{bc-ab}{abc}, \frac{ab-ac}{abc}.$$

2. Reduce to L. C. D. $\frac{x^3}{(x+y)^3}, \frac{y}{x+y}, \frac{xy}{(x+y)^2}$.

1. The L. C. Dd. = $(x+y)^3$.

2. $(x+y)^3 \div (x+y)^3, x+y, (x+y)^2 = 1, (x+y)^2, (x+y)$.

$$3. \frac{x^3}{(x+y)^3} \times \frac{1}{1}, \frac{y}{(x+y)} \times \frac{(x+y)^2}{(x+y)^2}, \frac{xy}{(x+y)^2} \times \frac{(x+y)}{(x+y)} \\ = \frac{x^3}{(x+y)^3}, \frac{y(x+y)^2}{(x+y)^3}, \frac{xy(x+y)}{(x+y)^3}.$$

3. Reduce to L. C. D. $\frac{1}{(a-c)(a-b)}, \frac{1}{(b-a)(b-c)},$

$$\frac{1}{(c-a)(c-b)}$$

Explanation.

Making the factors *like* in the arrangement of their letters, we have

$$\frac{1}{(a-c)(a-b)}, \frac{-1}{(a-b)(b-c)}, \frac{1}{(a-c)(b-c)}.$$

1. The L. C. Dd. = $(a-c)(a-b)(b-c)$.

2. The quotients are $b-c, a-c, a-b$.

3. The numerators with L. C. D. are

$$\frac{b-c, -(a-c), a-b}{(a-c)(a-b)(b-c)}.$$

EXERCISES.

Reduce to L. C. D. :

$$1. \ n, \frac{1+3n}{n-3}. \quad \left(n = \frac{n}{1}\right).$$

$$2. \ \frac{1}{x+2}, \frac{2}{x+3}.$$

$$3. \ \frac{2}{a^3x^2}, \frac{3}{ax^3}, \frac{4}{a^2x}.$$

$$4. \ \frac{1}{x^2-1}, \frac{3}{x+1}.$$

$$5. \ \frac{1}{a-x}, \frac{1}{a^2-x^2}.$$

$$6. \ \frac{x}{a-1}, \frac{y}{a^2-1}, \frac{z}{a^2-2a+1}.$$

$$7. \ \frac{a}{x^3y(x+y)}, \frac{b}{xy^3(x-y)}, \frac{c}{x^2y^2(x^2-y^2)}.$$

$$8. \ \frac{a-1}{a^2-2a-15}, \frac{a+2}{a^2+a-6}, \frac{a+3}{a^2-7a+10}.$$

$$9. \ \frac{a}{(a-b)(b-c)}, \frac{b}{(a-b)(c-b)}.$$

$$10. \ \frac{x+1}{x-1}, \frac{x^2+1}{x^2+x+1}, \frac{x^3}{x^3-1}.$$

Addition and Subtraction of Fractions.

RULE.

1. Reduce the fractions, when necessary, to their L. C. D.
2. Connect the numerators by the given signs of operation, + and -.
3. Write the result over the L. C. D. and simplify the fraction thus obtained.

1. Addition and subtraction of fractions are but special cases of simplification.

Simplify $\frac{a}{2} + \frac{b}{3} - \frac{a}{4}$.

$$\frac{a}{2} + \frac{b}{3} - \frac{a}{4}, \text{ reduced to L. C. D., } = \frac{6a + 4b - 3a}{12} = \frac{3a + 4b}{12}$$

EXERCISES.

Simplify :

$$1. \frac{2x}{5} + \frac{x}{3} - \frac{2x}{3}.$$

$$2. \frac{2x}{3} - x + \frac{3x}{5}.$$

$$3. \frac{7x-5}{8} - \frac{3x+2}{3} + \frac{x+1}{4}.$$

$$4. \frac{2x+3}{2x} + \frac{x+3}{4x} - \frac{18x+5}{8x^2}.$$

$$5. \frac{1-a}{1+a} + \frac{1}{1-a} - \frac{a^2}{1-a^2}.$$

$$6. \frac{1+x^2}{1-x^2} - \frac{1-x^2}{1+x^2}.$$

$$7. 1 + \frac{(a-b)^2}{4ab}.$$

$$8. \frac{x-1}{x-2} - \frac{x-3}{x-1}.$$

$$9. \frac{a}{a-1} - a + \frac{1}{a^2-a} + \frac{1}{a}.$$

$$10. \frac{bc}{(a-b)(a-c)} + \frac{bc}{(b-a)(b-c)} + \frac{ab}{(c-a)(c-b)}.$$

REVIEW.

1. m and n are the terms of a fraction ; with these terms write a negative fraction in various forms ; also a positive fraction.

2. ab and cd are the terms of a fraction ; with these terms write a positive fraction in various forms ; change the positive fractions to negative fractions.

3. Arrange in like order the letters in $\frac{1}{(a-b)(b-c)} + \frac{1}{(b-a)(a-c)} - \frac{1}{(c-a)(c-b)}$ without changing the value of the fractions.

4. Transform $\frac{2}{(x-2)(x-3)} - \frac{3}{(3-x)(4-x)} - \frac{1}{(x-4)(2-x)}$ so that the factors of the denominators will all begin with x .

5. Reduce to their simplest form :

$$1. \frac{18a^2c^2 - 3acf}{27ac^2 - 6ac^3}$$

$$6. \frac{x^6 + a^2x^3y}{x^6 - a^4y^2}$$

$$2. \frac{6x^2 - 7x - 20}{4x^3 - 27x + 5}$$

$$7. \frac{2x^2 + 3x + 1}{x^2 - x - 2}$$

$$3. \frac{x^2 + x - 2}{2x^2 - 3x + 1}$$

$$8. \frac{x^3 - 8x - 3}{x^4 - 7x^2 + 1}$$

$$4. \frac{x^4 - 14x^2 - 51}{x^4 - 2x^2 - 15}$$

$$9. \frac{x^4 - 1}{x^6 - 1}$$

$$5. \frac{x^{3m} + x^{2m} - 2}{x^{2m} + x^m - 2}$$

$$10. \frac{3x^4 - 14x^3 - 9x + 2}{2x^4 - 9x^3 - 14x + 3}$$

6. Transform the following improper fractions into integral or mixed quantities :

$$1. \frac{a^2}{a}; \frac{a+1}{a}; \frac{a-1}{a}; \frac{a^2-2b^2}{a-b}; \frac{2a^2+5}{a+3};$$

$$\frac{a^2+2ax+x^2}{ax}.$$

$$2. \frac{a^4-b^4}{a^2-2ab+b^2}; \frac{a^3-a^2-a+1}{a^2-2}; \frac{7x^2-18x+3}{x-4}$$

$$\frac{a^3-x^3+1}{a^2+ax-1}.$$

7. Reduce the following mixed quantities to fractional forms :

$$1. x - \frac{a^2-x^2}{x}.$$

$$6. ax + 4 - \frac{ax^2-y}{x+y}.$$

$$2. 5 + \frac{2x-7}{3x}.$$

$$7. 3 + \frac{3}{x^2-1}.$$

$$3. 1 - \frac{x-a-1}{a}.$$

$$8. a^2 - ax + x^2 + \frac{2x^3}{a+x}.$$

$$4. 1 + 2x - \frac{x-3}{5x}.$$

$$9. x + 1 - \frac{1-x^2}{x-1}.$$

$$5. 3x - 1 - \frac{x+a}{3a-2}.$$

$$10. x + 1 + \frac{1-x^2}{x-1}.$$

8. Simplify the following expressions :

$$1. \frac{a+b}{a-b} - \frac{a-b}{a+b}.$$

$$2. 1 - \frac{a}{a+b} + \frac{a}{a-b}.$$

$$3. a - y - \frac{a^2-y^2}{a+y}.$$

$$4. \frac{2a-4b}{2} + \frac{15a-4c}{12} - \frac{2a-b-c}{3}.$$

$$5. \frac{a}{b^2} + \frac{a^2+b}{ab^2} + \frac{1}{ab}.$$

$$6. \frac{a^4x^3 - a^3x^5}{x^4 - ax^2} + a^3x - a^4.$$

$$7. \frac{1}{(x+y)^2} - \frac{1}{(x-y)^2}.$$

$$8. 1 + \frac{a-b}{a+b} - \frac{a^2-b^2}{a^2-b^2}.$$

$$9. \frac{a^2}{b^3-a^3} + \frac{b^3a^2}{a^6-b^6}.$$

$$10. \frac{a^2-yz}{(x+y)(x+z)} + \frac{y^2-xz}{(y+z)(x+y)} + \frac{z^2-xy}{(y+z)(x+z)}.$$

Multiplication and Division of Fractions.

Laws affecting the Value of the Fraction.

Let $\frac{a}{b} = v$.

Then $a = bv$. *Why?*

And $am = bmv$. Ax. 3.

And $\frac{am}{b} = mv$. Ax. 4.

That is, *When the numerator is multiplied, the value of the fraction is multiplied.*

Consequently, *When the numerator is divided, the value of the fraction is divided.*

Again, since $\frac{am}{bm} = v$. (Page 77.)

$$\frac{a}{bm} = \frac{v}{m}. \quad \text{Ax. 4.}$$

That is, *When the denominator is multiplied, the value of the fraction is divided.*

Consequently, *When the denominator is divided, the value of the fraction is multiplied.*

1. Multiply $\frac{a}{b}$ by $\frac{c}{d}$.

Explanation.

Obviously, if c be taken alone as a multiplier, the result will be d times too great. $\frac{a}{b} \times c = \frac{ac}{b}$; but $\frac{ac}{b}$ is d times too great. $\frac{ac}{b} \div d = \frac{ac}{bd}$.

Hence $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$. That is, *The product of two fractions is the product of their numerators divided by the product of their denominators.*

2. Multiply $a + \frac{bx}{a}$ by $\frac{a}{d}$.

Process.

$$\left(a + \frac{bx}{a}\right) \times \frac{a}{d} = \frac{a^2 + bx}{a} \times \frac{a}{d} = \frac{a^2 + bx}{d}.$$

RULE.

Write the integers and mixed quantities in fractional form, indicate the multiplication required, cancel common factors, and find the product of the remaining factors of the numerators for a new numerator, and of the denominators for a new denominator.

3. Divide $\frac{a}{b}$ by $\frac{c}{d}$.

Explanation.

If c be taken alone as the divisor, the quotient will be d times too small.

$$\frac{a}{b} \div c = \frac{a}{bc}; \text{ but } \frac{a}{bc} \text{ is } d \text{ times too small.}$$

$$\frac{a}{bc} \times d = \frac{ad}{bc}. \text{ Hence } \frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}.$$

But $\frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$. That is, the quotient of two fractions is their product when the divisor is inverted.

4. Divide $\frac{x+1}{6}$ by $\frac{2x}{3}$.

Process.

$$\frac{x+1}{\underset{2}{\cancel{6}}} \times \frac{\overset{3}{\cancel{2}}}{2x} = \frac{x+1}{4x}.$$

RULE.

Invert the divisor and proceed as in multiplication.

EXAMPLES.

Simplify :

1. $\frac{x}{x-1} \div \frac{x}{2} \times \frac{a}{1-x}.$

Process.

$$\begin{aligned} \frac{x}{x-1} \times \frac{2}{x} \times \frac{a}{1-x} &= \frac{2a}{(x-1)(1-x)} = -\frac{2a}{(x-1)(x-1)} \\ &= -\frac{2a}{(x-1)^2}. \end{aligned}$$

2. $\left(b + \frac{bx}{a}\right) \times \frac{a}{x}.$

3. $\frac{x^2 - b^2}{bc} \times \frac{x^2 + b^2}{b+c}.$

4. $\frac{x-b}{8cd} \div \frac{3cx}{4d}.$

5. $\frac{x^4 - b^4}{x^2 - 2bx + b^2} \div \frac{a}{b^2 - x^2}.$

6. $\left(x - \frac{1}{x}\right) \div \left(1 + \frac{1}{x}\right).$

7. $\left(\frac{1}{a} + 1\right) \times \left(\frac{a^3}{1+a}\right).$

$$8. \left(\frac{a}{b} - \frac{b}{c} + \frac{c}{d} \right) \div \frac{a - b + c}{b^2 c^2 d^2}.$$

$$9. \frac{a^3 - x^3}{x^3} \div \frac{x^2 + ax + a^2}{a - x}.$$

$$10. \frac{(a + 1)^2}{2x} \times \frac{x^2}{a + 1}.$$

$$11. \frac{4x^2}{3y^2} \times \frac{12xy}{7z^2} \div \frac{8x^3}{35yz^2}.$$

$$12. \left(-\frac{2x}{3p} \right) \frac{7p^3}{77q^3} \left(-\frac{3pq^2}{35xy} \right).$$

$$13. \left(\frac{x}{y} + 1 \right) \div \left(\frac{x^2}{y} + \frac{y^2}{x} \right).$$

$$14. \frac{3(a - b)^2}{4(a + b)^2} \times \frac{7(a^2 - b^2)}{9(a - b)^3} \div \frac{14ab}{8(a + b)}.$$

$$15. \left(1 - \frac{y^4}{x^4} \right) \div \left(\frac{x^2 + y^2}{xy} \right).$$

$$16. \left(\frac{4x^2}{y^2} - 1 \right) \left(\frac{2x}{2x - y} - 1 \right).$$

$$17. \frac{a^3 - a^2y + ay^2}{x - 3} \times \frac{ax + xy - 3a - 3y}{a^3 + y^3}.$$

$$18. \left(m + \frac{mn}{m - n} \right) \left(n - \frac{mn}{m + n} \right).$$

$$19. \frac{x^2 + 3x + 2}{x^2 + 9x + 20} \times \frac{x^2 + 7x + 12}{x^2 + 5x + 6}.$$

$$20. \left(\frac{a^6 - x^6}{a^2 - 2ax + x^2} \times \frac{1}{a + x} \right) \div \left(\frac{a^2 + ax + x^2}{a - x} \times a^2 - ax + x^2 \right).$$

Complex Fractions.

A Complex Fraction has a fraction in one of its terms or in both.

1. Simplify the complex fraction $\frac{x^4 - \frac{1}{x^4}}{x + \frac{1}{x}}$.

Process.

$$\begin{aligned} \frac{x^4 - \frac{1}{x^4}}{x + \frac{1}{x}} &= \frac{\frac{x^8 - 1}{x^4}}{\frac{x^2 + 1}{x}} = \frac{x^8 - 1}{x^4} \times \frac{x}{x^2 + 1} = \frac{(x^4 + 1)(x^4 - 1)}{x^4} \times \\ &\frac{x}{x^2 + 1} = \frac{(x^4 + 1)(x^2 - 1)}{x^3} = \frac{x^6 - x^4 + x^2 - 1}{x^3} = \\ &x^3 - x + \frac{1}{x} - \frac{1}{x^3}. \quad \text{Ans.} \end{aligned}$$

NOTE.— $\frac{x^2 + 1}{x}$, being regarded a divisor, is inverted in the process of simplification.

2. Simplify $\frac{\frac{m}{m-n} - \frac{m}{m+n}}{\frac{n}{m-n} + \frac{m}{m+n}}$.

Explanation.

The L. C. D. of the denominators, $m - n$ and $m + n$, is $m^2 - n^2$.

Multiplying both terms of the complex fraction by $m^2 - n^2$, we have

$$\frac{m(m+n) - m(m-n)}{n(m+n) + m(m-n)} = \frac{m^2 + mn - m^2 + mn}{mn + n^2 + m^2 - mn} = \frac{2mn}{m^2 + n^2}. \quad \text{Ans.}$$

3. Simplify $\frac{3}{1 + \frac{3}{1 + \frac{3}{1-x}}}$, a continued fraction.

$$\begin{aligned} \frac{3}{1 + \frac{3}{1 + \frac{3}{1-x}}} &= \frac{3}{1 + \frac{3}{\frac{1-x+3}{1-x}}} = \frac{3}{1 + \frac{3}{4-x}} = \frac{3}{1 + \frac{3-3x}{4-x}} \\ &= \frac{3}{\frac{4-x+3-3x}{4-x}} = \frac{3}{\frac{7-4x}{4-x}} = \frac{12-3x}{7-4x}, \text{ Ans.} \end{aligned}$$

NOTE.—Begin by simplifying the mixed quantity at the foot of the complex fraction.

EXERCISES.

Simplify :

$$1. \frac{a + \frac{m}{n}}{b - \frac{c}{d}}$$

$$2. \frac{\frac{a}{4} + c}{x + \frac{z}{2}}$$

$$3. \frac{1 - \frac{3}{a}}{1 - \frac{1}{x} + \frac{1}{x^2}}$$

$$4. \frac{\frac{1}{3}(a-b)}{\frac{2}{3}a + \frac{3}{5}b}$$

$$5. \frac{a}{b + \frac{c}{d + \frac{e}{f}}}$$

$$6. \frac{\frac{1}{1+a} - \frac{1}{1-a}}{\frac{1}{1-a} + \frac{1}{1+a}}$$

$$7. \frac{\frac{x^3+y^3}{x^2-y^2}}{x-y}$$

$$8. \frac{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}}{\frac{a}{b} + \frac{b}{c} + \frac{a}{c}}$$

$$9. \frac{\frac{a}{a+b} + \frac{b}{a-b}}{\frac{a}{a-b} - \frac{b}{a+b}}.$$

$$10. \frac{\frac{1+x^2}{1-x^2} - \frac{1-x^2}{1+x^2}}{\frac{1}{1+x} - \frac{1}{1-x}}.$$

MISCELLANEOUS EXERCISES.

Reduce to Lowest Terms :

$$1. \frac{4m^2n^2}{2m^2n + 2mn^2}.$$

$$4. \frac{1-a}{1-a}.$$

$$2. \frac{a^3 - x^3}{a^2 - x^2}.$$

$$5. \frac{6a^2b^2 - b^2}{2b^2 - b^3}.$$

$$3. \frac{x^2 + 7x + 6}{x^2 + 5x + 4}.$$

$$6. \frac{a^4 - m^4}{a^3 - a^2m - am^2 + m^3}.$$

Change to whole or mixed quantities :

$$1. \frac{a^4 - b^4}{a^2 - 2ab + b^2}.$$

$$3. \frac{7x^2 - 18x + 3}{x - 4}.$$

$$2. \frac{a^3 - a^2 - a + 1}{a^2 - 2}.$$

$$4. \frac{a^3 - x^3 + 1}{a^2 + ax - 1}.$$

$$5. \frac{x^3 + ax^2 - 3a^2x - 3a^3}{x - 2a}.$$

Change to improper fractions :

$$1. a + b - \frac{a^3 - b^3}{a^2 - ab + b^2}.$$

$$2. \frac{a^2 + b^2 + c^2}{2bc} - 1.$$

$$3. 1 - \frac{a^2 - b^2 + c^2}{2b}.$$

$$4. (a-1)^4 - \frac{(a-1)^4}{a}.$$

$$5. x^3 + 3xy + y^3 + \frac{9x^2y^2}{x^3 - 3xy + y^3}.$$

$$6. x - 1 - \frac{x - 1}{x^2 + x + 1}.$$

$$7. x^2 - \left[1 - \left(1 - \frac{2}{x + 1} \right) \right].$$

Express with the L. C. D. :

$$1. \frac{1}{a(x-y)}, \frac{1}{b(x-y)}, \frac{ab}{x^2 - y^2}.$$

$$2. -\frac{1}{a-c}, -\frac{1}{c-a}.$$

$$3. \frac{a}{x-y}, \frac{-a}{y-x}.$$

$$4. \frac{1}{1-x+x^2}, \frac{-1}{x-1-x^2}.$$

$$5. -\frac{1-a}{a^3-1}, \frac{-(1-a)}{1-a^3}.$$

$$6. 1-a, \frac{a-b}{x}, \frac{b}{a-x}.$$

$$7. \frac{x+1}{x(x-2)}, \frac{3x+2}{x(x+2)}, \frac{2x-1}{x^2-x-2}.$$

Simplify :

$$1. \frac{a-b}{c-d} \text{ of } \frac{a+b}{c+d} \text{ of } \frac{c^2-d^2}{a^2-b^2}. \quad \text{"Of" is equivalent to the sign } \times.$$

$$2. -\frac{a}{b} \div \frac{a}{b}.$$

$$3. \frac{a-c}{a-b} \div \frac{a+b}{a+c}.$$

$$4. \frac{\frac{a}{b}}{\frac{c}{b}}$$

$$7. \frac{1 - \frac{x}{y}}{\frac{y}{x}}$$

$$5. \frac{\frac{a+b}{a-b}}{\frac{a-c}{a+c}}$$

$$8. \frac{\frac{a-b-c}{b}}{\frac{a+1}{b}}$$

$$6. \frac{1 + \frac{a}{b}}{1 - \frac{a}{b}}$$

$$9. \left(1 - \frac{a}{1}\right) \div \left(\frac{1}{a} - 1\right).$$

$$10. \left(\frac{1}{a} + \frac{1}{ab^3}\right) \div \left(b + \frac{1}{b} - 1\right).$$

$$11. \frac{1}{2a-1 + \frac{1}{1 + \frac{a}{1-a}}} \times \frac{a}{2}.$$

$$12. \frac{\frac{1}{x} + \frac{1}{y} + \frac{1}{z}}{\frac{x}{y} + \frac{y}{z} + \frac{z}{x}}.$$

Substitute :

1. $(y+3)$ for x in $x^3 + 2x^2 - x - 10$.

2. $(m-n)$ for y in $y^2 + ny + x$.

3. n for p in $n^4 + 2n^3p + 2n^2p^2 + 2np^3 + p^4$.

4. $(a+1)$ for s , and $(a-1)$ for t in $s^3 + ts^2 + t^2s + t^3$.

5. $\frac{a+1}{ab+1}$ for x , and $\frac{ab+a}{ab+1}$ for y , in $\frac{x+y-1}{x-y-1}$.

Perform the division indicated, restrict the quotient to five terms, and make the last term fractional.

1. $x \div (1 + x)$.

4. $1 \div (1 - x + x^2)$.

2. $a \div (a - b)$.

5. $x^2 \div (x + 3)$.

3. $(1 + x) \div (1 - x)$.

Show that : $x^2 - xy + y^2 - \frac{2y^3}{x + y} = \frac{x^3 - y^3}{x + y}$;

that $\frac{a^2 + b^2}{(a - b)^3} \div \frac{a^2 + b^2}{a^2 - b^2} \times \frac{a^2 - b^2}{(a + b)^2} \times \frac{a^2 - b^2}{a + b} = 1$;

and that $\frac{m - 2}{2m - 14} \times \frac{m^2 - 5m - 14}{2m^2 + 4m} \times \frac{4m^2}{m^2 - 4} \div \frac{2mn - 14n}{m^2 - 5m - 14}$
 $= \frac{m}{2n}$.

Infinity.

Dividing a by b , we have for quotient $\frac{a}{b}$.

Now, if while the value of a remains fixed, the value of b constantly increases, the value of the fraction $\frac{a}{b}$ constantly diminishes.

When the value of b becomes exceedingly great, the value of $\frac{a}{b}$ becomes exceedingly small.

When the value of b becomes greater than any assignable quantity, the value of $\frac{a}{b}$ becomes less than any assignable quantity ; that is, the value of the fraction is practically 0. Hence, b having become algebraically infinite, is represented by the symbol ∞ and we have $\frac{a}{\infty} = 0$.

If, on the contrary, while the value of a remains fixed the value of b constantly diminishes, the value of the fraction constantly increases.

When the value of b becomes less than any assignable quantity, the value of the fraction becomes greater than any assignable quantity, is said to be infinite, and is represented by the symbol ∞ . Hence we have $\frac{a}{0} = \infty$.

The Symbol $\frac{0}{0}$.

When a factor is 0, the product is 0. Therefore,

$$1. a \times 0 = 0.$$

$$2. \infty \times 0 = 0.$$

$$3. 0 \times 0 = 0.$$

Dividing each of these equations by 0, we have,

$$1. a = \frac{0}{0}.$$

$$2. \infty = \frac{0}{0}.$$

$$3. 0 = \frac{0}{0}.$$

Hence the symbol $\frac{0}{0}$ may represent a *finite quantity, infinity, or zero*.

Since $\frac{0}{0} = a$, and a may have an infinite number of values, $\frac{0}{0}$ is called the *Symbol of Indetermination*.

The Value of a^0 .

Since $a^1 = a$, dividing by a we have

$$a^{1-1} = 1. \quad (\text{Ax. 4 and Law of Exponents.})$$

That is, $a^0 = 1$.

EQUATIONS.

When two expressions are equal in value, the algebraic statement of their equality is called an **Equation**.

The parts separated by the sign of equality are called the *Members of the Equation*.

The expression on the left of the sign of equality is called the *First Member* of the equation ; the one on the right of the sign of equality is called the *Second Member*.

Equations are classified as :

1. *Numerical*, the known quantities being numerals.
2. *Literal*, the known quantities, or some of them, being letters.
3. *Identical*, the members being equal for any values assigned the letters.
4. Of the *First Degree*, *Second Degree*, *Third Degree*, etc., accordingly as the highest power of the unknown quantity is denoted by 1, 2, 3, etc.

An equation of the First Degree is also called a **Linear Equation** ; more commonly, a **Simple Equation**.

The value of the unknown quantity of an equation is called its **Root**. The root substituted for the unknown quantity renders the equation an identity.

The root of the equation, $3x - 4 = x + 6$, is 5 ; for 5, substituted for x , gives us the identity, $15 - 4 = 5 + 6$, or $11 = 11$.

The production of identity *verifies* the root, and *satisfies* the equation.

The solution of an equation consists in finding the value of the unknown quantity ; and the value of the unknown quantity is found by transforming or reducing the equation to

an *equivalent* equation, or through a series of equivalent equations.

Equivalent equations result from the application of one or more of the following principles :

1. Any term (or all the terms) of an equation may be transposed from one member to the other.

2. All the positive signs of an equation may be changed to negative signs, and all the negative signs to positive signs.

[This is, in effect, transposing all the terms.]

3. Both members of an equation may be multiplied or divided by the same quantity, except 0.

4. When expedient, operations may be indicated, and indicated operations may be performed.

Illustrations.

1. Let the equation be $x - a = 8$.

Transposing, we have the equivalent equation, $x = 8 + a$, which exhibits the value sought.

2. Let the equation be $x + \frac{2x}{3} = 3\frac{1}{3}$.

Explanation.

The equivalent equations are :

By principle third, $3x + 2x = 10$.

By principle fourth, $5x = 10$.

By principle third, $x = 2$.

An Equation of Simple Form.

1. Solve $5x + 7 = 11 + 4x$.

1. By transposing [Principle 1], we have $5x - 4x = 11 - 7$.

2. By performing operations indicated, $x = 4$, the root.

To test the accuracy of a result is to *satisfy the equation*, or *to verify the root*.

Substituting 4 for x in the given equation, we have the identity $20 + 7 = 11 + 16$.

Why is $5x - 4x = 11 - 7$ an equivalent equation?

Substituting 4 for x , we have the identity $20 - 16 = 11 - 7$.

The identities already produced verify the root.

An Identical Equation.

Prove the identity $\frac{a^3 - 1}{a - 1} = a^2 + a + 1$.

Let $a = 5$.

Then $\frac{5^3 - 1}{5 - 1} = 5^2 + 5 + 1$,

and $\frac{125 - 1}{5 - 1} = 25 + 5 + 1$;

That is $\frac{124}{4} = 31$.

Or,

$$\begin{array}{r} a - 1 \overline{) a^3 - 1} \quad (a^2 + a + 1 \\ \underline{a^3 - a^2} \\ a^2 - 1 \\ \underline{a^2 - a} \\ a - 1 \\ \underline{a - 1} \\ 0 \end{array}$$

A Fractional Equation.

Solve $\frac{1}{2}x + \frac{3}{4} = \frac{5}{8}x - \frac{1}{4}$.

Explanation.

The L. C. Dividend of the denominators is 8.

Multiplying by 8, we have $4x + 6 = 5x - 2$.

Transposing, we have $4x - 5x = -6 - 2$.

Performing operation indicated, we have $-x = -8$.

Changing all the signs (Principle 2), we have $x = 8$, the root.

How is the equation satisfied?

NOTE.—Multiplying by the L. C. Dd. of the denominators is called “clearing of fractions.”

An Equation containing Decimal Fractions.

Solve $.4x - .075x = 12 - .275x$.

Explanation.

The L. C. Dd. of the denominators is 1000.

Multiplying by 1000, $400x - 75x = 12000 - 275x$.Transposing, $400x - 75x + 275x = 12000$.Removing $-$ and $+$, $600x = 12000$.Dividing by 600, $x = 20$, the root.*How is the root verified?*

A Literal Equation.

Solve $\frac{a}{x} + \frac{b}{c} - \frac{a}{c} = 0$.

Explanation.

The L. C. Dd. of the denominators is xc .Multiplying by xc , $ac + bx - ax = 0$.Transposing, $bx - ax = -ac$.Changing signs, $ax - bx = ac$.Factoring, $x(a - b) = ac$.Dividing by $a - b$, $x = \frac{ac}{a - b}$, the root.*How is the equation satisfied?*

EXERCISES.

1. Prove the following identities :

1. $(x^2 + 3x + 2) \div (x + 1) = x + 2$.

2. $\frac{8x^3 - y^3}{2xy + 4x^2 + y^2} = 2x - y$.

3. $(a - x)(x - a)(1 - a)(a - 1) = (a^2 + x^2 - 2ax)(a - 1)^2$.

4. $(a^0 \times a^{-1} \times b^0 \times b^{-1} \times c^0 \times c^{-1})abc = 1$.

5. $\frac{a^2}{a^2b^0} \times \frac{n+1}{n-1} \times \frac{b^0}{n+1} \times \frac{1}{(n-1)^{-1}} = 1$.

2. Solve :

1. $5x - 7 = 4x + 3.$
2. $6x + 1 = 21 + x.$
3. $a - x = 0.$
4. $5x + 22 - 2x = 31.$
5. $3x - 20 = -x - 4.$
6. $x - 1 = 4x - 91.$
7. $3x - 4x + 10x + 5x = 35.$
8. $15x - 2(5x - 4) - 39 = 0.$
9. $6(23 - x) - 3x = 3(4x - 27).$
10. $x - [3 + \{x - (3 + x)\}] = 5.$

3. Solve :

1. $2x - \frac{1}{2}x + 1 = 5x - 2.$
2. $x + \frac{x}{2} + \frac{x}{3} = -11.$
3. $\frac{2x}{5} - \frac{9x}{20} - \frac{7x}{10} = \frac{5}{4}.$
4. $x - \frac{x}{4} + 25 = \frac{x}{3} + \frac{x}{2} + 21.$
5. $\frac{2x - 1}{3} - \frac{13}{42} = \frac{5x - 4}{6}.$
6. $\frac{x}{2} + \frac{x}{3} + \frac{x}{5} = \frac{x}{4} + \frac{47}{6}.$
7. $\frac{3}{x} = \frac{3}{5}.$
8. $\frac{6x - 4}{3} - 2 = \frac{18 - 4x}{3} + x.$
9. $\frac{6x + 7}{9} + \frac{7x - 13}{27} = \frac{2x + 4}{3}.$
10. $\frac{3x}{5} + 2\frac{1}{2} + 11 = \frac{x}{4} + 17.$

4. Solve:

1. $.2x - 3.5 = .6x + 1.5$.
2. $.4x - .075x = 12 - .275x$.
3. $.375x - 1.875 = .12x + 1.185$.
4. $.6x - 1.5x = .2 - .15x$.
5. $.125x - .0625x = .375$.

5. Solve:

1. $ay + 2b = 3by + 4a$.
2. $ab - 3cx - 2ax - c = 0$.
3. $2a - x = -4x + 8a$.
4. $am - ax = -am + 2ax$.
5. $5x - 2a - 3b + x = -x - 4a + 4b - 12a$.
6. $am - 5x = bc - ax$.
7. $abc - a^2x = ax - a^2b$.
8. $3acx - 6bcd = 12cdx + abc$.
9. $x^2 - a^2 = x - a$.
10. $(a + x)(b + x) = x(x - c)$.

Some Special Cases.

1. Solve $(x - 3)^2 - (5 - x)^2 = -4x$.

Explanation.

Squaring as indicated, $x^2 - 6x + 9 - 25 + 10x - x^2 = -4x$.

Transposing, $x^2 - x^2 - 6x + 10x + 4x = +25 - 9$.

Performing operations indicated, $8x = 16$.

Dividing by 8, $x = 2$, the root.

EXERCISES.

Solve:

1. $x^2 + a^2 = (b - x)^2$.
2. $(x + 2)^2 - (x - 1)^2 = 5(2x + 3)$.
3. $(x + 1)^2 + (x - 5)^2 = 2(x + 5)^2$.
4. $x^2 + 4a^2 + a^4 = (x + a^2)^2$.
5. $(x - 1)^3 + x^3 + (x + 1)^3 = 3x(x^2 - 1)$.

2. Solve $7x - \frac{11x - 3}{4} = 3x + 7$.

Explanation.

Multiplying by 4, $28x - (11x - 3) = 12x + 28$.

Removing parenthesis, $28x - 11x + 3 = 12x + 28$.

Transposing, $28x - 11x - 12x = 28 - 3$.

Performing operations indicated, $5x = 25$.

Dividing by 5, $x = 5$, the root.

In clearing of fractions notice the treatment of the numerator of a negative fraction.

EXERCISES.

Solve:

1. $\frac{4(x + 2)}{5} = 7 - 5x$.

2. $2x - 8 - \frac{24 - 2x}{7} = 0$.

3. $\frac{4}{x} + 3 = 0$.

4. $\frac{ax}{c} = b - x$.

5. $x - (3x - 4) - \frac{5 - 2x}{4} = 2$.

3. Solve $\frac{x - 8}{7} + \frac{x - 3}{3} + \frac{5}{21} = 0$.

1. Multiplying by 21 (L. C. Dd.), $3x - 24 + 7x - 21 + 5 = 0$.

2. Transposing, $3x + 7x = 24 + 21 - 5$.

3. Performing operations indicated, $10x = 40$.

4. Dividing by 10, $x = 40$, the root.

EXERCISES.

Solve:

1. $4 - \frac{x - 9}{8} = \frac{x}{22} - \frac{1}{2}$.

$$2. \frac{8-5x}{12} + \frac{5x-6}{4} - \frac{7x+5}{6} = 0.$$

$$3. \frac{x+3}{4} - \frac{x-8}{5} - \frac{x-5}{2} = -1.$$

$$4. \frac{ax-b}{c} - \frac{bx+c}{a} = abc.$$

$$5. \frac{1}{3}(x-3) - \frac{1}{4}(x-8) + \frac{1}{5}(x-5) = 0.$$

$$4. \text{ Solve } \frac{x+3}{x-1} - \frac{x-4}{x+1} = \frac{2x}{x^2-1}.$$

Explanation.

1. The L. C. Dd. of the denominators is $x^2 - 1$.

2. $(x^2 - 1) \div \left(\begin{array}{c} x-1 \\ x+1 \\ x^2-1 \end{array} \right) = \begin{array}{c} x+1 \\ x-1 \\ 1 \end{array} \}$ multipliers of the numerators.

3. Clearing of fractions, $(x+3)(x+1) - (x-4)(x-1) = 2x \times 1$.

4. Removing parentheses, $x^2 + 4x + 3 - x^2 + 5x - 4 = 2x$.

5. Suppressing x^2 and transposing, $4x + 5x - 2x = 4 - 3$.

6. Performing operation indicated, $7x = 1$.

7. Dividing by 7, $x = \frac{1}{7}$, the root.

EXERCISES.

Solve:

$$1. \frac{3}{1-x} - \frac{2}{1+x} - \frac{1}{1-x^2} = 0.$$

$$2. \frac{4}{1+x} + \frac{x+1}{1-x} - \frac{x^2-3}{1-x^2} = 0.$$

$$3. \frac{a}{x+b} - \frac{b}{x-b} = \frac{a^2-b^2}{x^2-b^2}.$$

$$4. \frac{6}{x-1} - \frac{4x}{x^2-1} = \frac{3}{x+1}.$$

$$5. b + \frac{x}{1+\frac{1}{b}} = 1 + \frac{x}{1-\frac{1}{b^2}}.$$

5. Solve $\frac{3x + 14}{21} - \frac{x + 6}{3x - 9} = \frac{x}{7}$.

Explanation.

1. $\frac{x}{7} = \frac{3x}{21}$.

2. Suppressing $\frac{3x}{21}$ in both members, we have $\frac{14}{21} \left(\text{or } \frac{2}{3} \right) - \frac{x + 6}{3x - 9} = 0$.

3. Transposing and changing signs, $\frac{x + 6}{3x - 9} = \frac{2}{3}$.

4. Clearing of fractions, $3x + 18 = 6x - 18$.

5. Transposing, $3x = + 36$.

6. Dividing by 3, $x = 12$, the root.

EXERCISES.

Solve :

1. $\frac{8x + 5}{14} + \frac{7x - 3}{6x + 2} = \frac{4x + 6}{7}$.

2. $\frac{9x + 20}{36} = \frac{4x - 12}{5x - 4} + \frac{x}{4}$.

3. $\frac{6x + 7}{9} + \frac{7x - 13}{6x + 3} = \frac{2x + 4}{3}$.

4. $2\frac{1}{2} - \frac{3}{2x + 4} = \frac{2x - 1}{4} - \frac{x}{2}$.

5. $\frac{10x + 17}{18} - \frac{12x + 2}{11x - 8} = \frac{5x - 4}{9}$.

6. Solve $\frac{x - 1}{x - 2} - \frac{x - 3}{x - 4} = \frac{x - 2}{x - 3} - \frac{x - 4}{x - 5}$.

Explanation.

1. Reducing each member to a common denominator, and performing the subtraction indicated, we have $\frac{-2}{(x-2)(x-4)} = \frac{-2}{(x-3)(x-5)}$.

2. Since the fractions are equal, and the numerators are equal, the denominators are equal, and we have $(x-2)(x-4) = (x-3)(x-5)$.

3. Removing parentheses, $x^2 - 6x + 8 = x^2 - 8x + 15$.
4. Transposing and simplifying, $2x = 7$.
5. Dividing by 2, $x = 3\frac{1}{2}$, the root.

EXERCISES.

Solve :

$$1. \frac{x}{x-2} - \frac{x+1}{x-1} = \frac{x-8}{x-6} - \frac{x-9}{x-7}.$$

$$2. \frac{x+5}{x+4} - \frac{x-6}{x-7} = \frac{x-4}{x-5} - \frac{x-15}{x-16}.$$

$$3. \frac{x-7}{x-9} - \frac{x-9}{x-11} = \frac{x-13}{x-15} - \frac{x-15}{x-17}.$$

$$4. \frac{x+3}{x+6} - \frac{x+6}{x+9} = \frac{x+2}{x+5} - \frac{x+5}{x+8}.$$

$$5. \frac{x+2}{x} + \frac{x-7}{x-5} - \frac{x+3}{x+1} = \frac{x-6}{x-4}.$$

$$7. \text{ Solve } \frac{x-8}{x-10} + \frac{x-4}{x-6} = \frac{x-5}{x-7} + \frac{x-7}{x-9}.$$

Explanation.

1. Transposing in order to have the second fraction in each member negative, we write $\frac{x-8}{x-10} - \frac{x-5}{x-7} = \frac{x-7}{x-9} - \frac{x-4}{x-6}$.

2. Simplifying each member as in the previous example, we have,

$$\frac{6}{(x-10)(x-7)} = \frac{6}{(x-9)(x-6)}.$$

Hence, $(x-10)(x-7) = (x-9)(x-6)$,

and $x^2 - 17x + 70 = x^2 - 15x + 54$.

$$2x = 16.$$

$x = 8$, the root.

EXERCISES.

Solve:

$$1. \frac{4x-17}{x-4} + \frac{10x-13}{2x-3} = \frac{8x-30}{2x-7} + \frac{5x-4}{x-1}.$$

$$2. \frac{x-1}{x-2} + \frac{x-6}{x-7} = \frac{x-5}{x-6} + \frac{x-2}{x-3}.$$

$$3. \frac{x}{x-2} + \frac{x-9}{x-7} = \frac{x+1}{x-1} + \frac{x-8}{x-6}.$$

$$4. \frac{x-4}{x-5} + \frac{x-8}{x-9} = \frac{x-5}{x-6} + \frac{x-7}{x-8}.$$

$$5. \frac{x^2-5}{x^2-6} + \frac{x^2-11}{x^2-12} = \frac{x^2-7}{x^2-8} + \frac{x^2-9}{x^2-10}.$$

$$8. \text{ Solve } \frac{1}{a(b-x)} + \frac{1}{b(c-x)} = \frac{1}{a(c-x)}.$$

Explanation.

The L. C. Dd. of the denominators is $ab(b-x)(c-x)$.

Hence, we have 1. $b(c-x) + a(b-x) = b(b-x)$.

$$2. bc - bx + ab - ax = b^2 - bx.$$

$$3. -ax = b^2 - ab - bc.$$

$$4. x = -\frac{b}{a}(b-a-c).$$

$$5. x = \frac{b}{a}(a-b+c).$$

$$9. \text{ Solve } \frac{x-a}{b-x} = \frac{x-b}{a-x}.$$

Explanation.

Changing the signs of both terms of $\frac{x-b}{a-x}$, we have

$$1. \frac{x-a}{b-x} = \frac{b-x}{x-a}.$$

$$2. (x-a)^2 = (b-x)^2.$$

$$3. x^2 - 2ax + a^2 = b^2 - 2bx + x^2.$$

$$4. 2bx - 2ax = b^2 - a^2.$$

$$5. x(2b-2a) = b^2 - a^2.$$

$$6. x = \frac{b^2 - a^2}{2(b-a)} = \frac{1}{2}(b+a).$$

$$10. \text{ Solve } \frac{x+a}{x-a} = \frac{x-a}{x+a}.$$

Explanation.

Reducing each member to a mixed quantity,

$$1 + \frac{2a}{x-a} = 1 - \frac{2a}{x+a}.$$

Suppressing the 1's, and dividing by $2a$, $\frac{1}{x-a} = \frac{-1}{x+a}$.

Clearing of fractions,

$$x+a = a-x.$$

$$2x = a-a=0.$$

$$x=0.$$

EXERCISES.

Solve :

$$1. \ x(x-5) = x^2 - 30.$$

$$2. \ (x+1)(x-3) = x^2 - 3x + 1.$$

$$3. \ 2x^2 = (x+1)^2 + (x+3)^2.$$

$$4. \ (a+x)(b+x) - (c-x)(d-x) = 0.$$

$$5. \ (x+1)^2 + (x-5)^2 = 2(x+5)^2.$$

$$6. \ b(2x-a) - a^2 = 2x(a+b) - 3ab.$$

$$7. \ ax - bx = c + dx - m.$$

$$8. \ (a-b)(x-a) = (a-c)(x-b).$$

$$9. \ \frac{1}{3} \{4a(1+x) - \frac{2}{4}(a+ax)\} = 0.$$

$$10. \ (x-a-b)^2 - (x-a)(x-b) + ab = 0.$$

$$11. \ \frac{106x+7}{11x+5} - \frac{x+1}{15} + \frac{2x-1}{30} = \frac{597}{30}.$$

$$12. \ \frac{x-3}{x-4} - \frac{x-4}{x-5} = \frac{x-5}{x-6} - \frac{x-6}{x-7}.$$

$$13. \ \frac{a^2-x}{c} - \frac{b^2-x}{a} - \frac{c^2-x}{b} = \frac{a^2}{c} - \frac{b^2}{a}.$$

$$14. \ \frac{(x+1)^2}{(x+2)^2} = \frac{x-4}{x-2}.$$

$$15. \ \frac{ax+b}{ax-b} - \frac{3b}{ax+b} = \frac{a^2x^2+b^2}{a^2x^2-b^2}.$$

$$16. \frac{1}{4}x(x-a) - \left(\frac{x+a}{2}\right)^2 = \frac{2a}{3}\left(x - \frac{a}{2}\right).$$

$$17. \frac{2x+1}{29} - \frac{402-3x}{12} = 9 - \frac{471-6x}{2}.$$

$$18. \frac{4+x}{4} - \frac{1-x}{7} - \frac{1}{5}(8-x) = \frac{x-23}{5} + 7.$$

$$19. 5 - x\left(3\frac{1}{2} - \frac{2}{x}\right) = \frac{1}{2}x - \frac{3x - (4-5x)}{4}.$$

$$20. \frac{ax+b}{c} + \frac{ax+b}{cx+b} = \frac{2ax+d}{2c} + \frac{b}{c}.$$

REVIEW.

Factor:

1. $2x^2 - xy - 10y^2.$

6. $x^6 - 1.$

2. $x^6 - y^{12}.$

7. $a^2x^2 - 4a^2x + 4a^2.$

3. $1 - a^8.$

8. $x^2 - y^2 + x - y.$

4. $4a^{2m} - 4a^mb^n + b^{2n}.$

9. $x^2 - 3x^2y + 3xy^2 - y^2.$

5. $a^4 + a^2b^2 + b^4.$

10. $3a^2 + a - 2.$

Simplify:

1. $3x - [2y - (x+y) - \{-y - (y - \overline{x-y})\}].$

2. $\left(\frac{a^2 - ax}{a^2 - x^2} \div \frac{a^2 - 2ax + x^2}{a^2 - x^2}\right)\left(1 - \frac{x^2}{a^2}\right).$

3. $\frac{\frac{a^2+b^2}{b} - a}{\frac{1}{a} - \frac{1}{b}} \left(\frac{a^2 - b^2}{a^3 + b^3}\right).$

4. $a - [5b - \{a - (5c - \overline{2c - b} - 4b) + 2a\}].$

5. $x - [5y - \{x - (5z - \overline{2z - y}) + 2x - (x - 2y - z)\}].$

Solve:

1. $\frac{x+6}{11} - \frac{2x-18}{3} + \frac{2x+3}{4} = 5\frac{1}{3} + \frac{3x+4}{12}.$

2. $\frac{2x+a}{b} - \frac{2a-x}{a} = \frac{3x-b}{b}.$
3. $\frac{a^2c}{b} + x^2 = (a+x)(b+x) - a(b+c).$
4. $x - \left(\frac{1-x}{4} - \frac{1+x}{2} \right) = \frac{x+3}{2}.$
5. $\frac{ab+x}{b^2} - \frac{b^2-x}{a^2b} = \frac{x-b}{a^2} - \frac{ab-x}{b^2}.$
6. $\frac{(a^2x+2b)}{a-2b} = \frac{a^2x}{a-2b} - \frac{4b^2}{x}.$
7. $\frac{5x+2}{3} - \left(3 - \frac{3x-1}{2} \right) = \frac{3x+19}{2} - \left(\frac{x+1}{6} + 3 \right).$
8. $\frac{x}{2a} - 3 + \frac{x}{4a^3} = \frac{x}{3a^2} - 2a(2-3a).$
9. $\frac{x+a}{x+b} + \frac{x+b}{x+c} = 2.$
10. $\frac{m(3a^2-2b^2)}{a+b} = \frac{2a^2-b^2}{ax+bx} + \frac{a-b}{x}.$

PROBLEMS.

A problem in Algebra proposes a question for solution by an equation.

The equation, being a statement of equality, can be written only after we have pondered the problem and discovered wherein the equality exists.

The existing equality may be stated *explicitly*; as in the following question:

1. What is the number whose one-half, one-third, and one-fourth, increased by 25, equals 220?

Explanation.

1. Letting x represent the number, and expressing the problem in algebraic language, we have the equation, $\frac{x}{2} + \frac{x}{3} + \frac{x}{4} + 25 = 220$.

2. Solving by equivalent equations, we have

$$6x + 4x + 3x + 300 = 2640.$$

$$13x = 2340.$$

$$x = 180, \text{ the number required.}$$

The existing equality may be stated *implicitly*, and must be found by deduction, as in the following question :

2. Out of a cask of wine a third part having leaked away, 21 gallons were afterward drawn, and the cask was then half full. How much did the cask hold ?

Explanation.

1. Letting x represent the number of gallons originally in the cask, and denoting algebraically the sum of what leaked away and was drawn, we have $\frac{x}{3} + 21$.

Now, since the cask was half full, and still contained $\frac{x}{2}$ gallons, the plain *implication* is that the part that leaked away and was drawn also equals $\frac{x}{2}$. Hence, we have the equation $\frac{x}{3} + 21 = \frac{x}{2}$.

2. Solving by equivalent equations, we have

$$2x + 126 = 3x$$

$$3x - 2x = 126$$

$$x = 126 \text{ gallons, the answer required}$$

PROBLEMS.

1. What number is it whose third part exceeds its fourth part by 16 ?

$$\frac{x}{3} - \frac{x}{4} = 16.$$

2. B. and C. buy an automobile for \$1000. If B. pays three times as much as C., how much does each pay ?

3. Carl and his brother have \$24, and Carl has \$8 more than his brother. How much money has each?

4. What is the number whose double exceeds its half by 150?

5. The half and fifth of a number equal 56. Find the number.

6. A father is three times as old as his son, and the difference of their ages is 24 years. What is the age of each?

7. What number exceeds the sum of its third, tenth, and twelfth parts by 87?

$$x - \left(\frac{x}{3} + \frac{x}{10} + \frac{x}{12} \right) = 87.$$

8. A man sold a horse for \$35 and half as much as he gave for it, and gained thereby \$10. How much did he give for the horse?

9. \$2000 is divided between A. and B. B. receives \$3 as often as A. receives \$2. How much does each receive?

If $2x$ represents A.'s share, what will represent B.'s share?

10. My horse and saddle are worth \$132, and the horse is worth ten times as much as the saddle. What is the value of the horse?

11. Divide the number 70 into two parts, such that three times the smaller shall be equal to twice the larger.

12. Ross is now twice as old as Raymond, and eight years ago he was six times as old. What is the present age of each?

If the age of Raymond is x now, what is the age of Ross? If the age of Raymond was $x - 8$ eight years ago, what was the age of Ross then?

13. A man is now twice as old as his son; 15 years ago he was three times as old as his son. What is the age of each?

14. Divide 42 into two parts so related that one-fifth of the one plus one-sixth of the other shall equal 8.

15. If two men going into business invest \$2600, and one invests four-ninths as much as the other, how much does each invest?

16. The sum of three numbers is 72. The second is double the first, and the third equals the first plus the second. Find the three numbers.

17. A. is 16 years older than B., and his age is as far above 50 as B.'s age is below 40. Find their ages.

18. The width of a room is two-thirds of its length. Another room having 3 feet more of width and 3 feet less of length, is a square. Find the dimensions of each room.

19. A father is 48 years old, and his son is 12 years old. In how many years will the father's age be double the son's age?

20. A post is $\frac{1}{4}$ in the mud, $\frac{1}{8}$ in the water, and 10 feet above the water. What is the whole length of the post?

21. The sum of two numbers is 20, and their difference is 6. Required the numbers.

x and $x + 6$ are the numbers.

22. The difference of the squares of two consecutive numbers is 41. Find the numbers.

If x is one of the numbers, what is the other?

23. Divide 150 dollars among 6 men, 10 women, and 40 children, and give each man one dollar more than each woman, and each child half as much as each woman.

24. A. can do a piece of work in 6 days and B. in 8 days. Working together, in how many days can they do the work?

$$\frac{1}{6} + \frac{1}{8} = \frac{1}{x}.$$

25. Mary and Martha have the same amount of money. If Mary should give Martha \$40, she would have one-third as much as Martha. How much money has each?

26. A debtor owed \$900 : to A. a certain sum, to B. twice that sum, and to C. twice as much as to A. and B. How much did he owe to each?

27. Divide 410 into four parts, making the second one-half of the first, the third one-third of the second, and the fourth one-fourth of the third.

28. Divide \$1285 among A., B., and C., making A.'s share \$25 more than five-sixths of B.'s share, and making C.'s four-fifteenths as much as B.'s.

29. A front wheel of a wagon has made 10 revolutions more than the corresponding hind wheel. The circumference of these wheels being respectively 2 and 5 yards, find how far the wagon has moved.

30. A fish was caught whose tail weighed 9 lbs., his head weighed as much as his tail and half his body; his body weighed as much as his head and tail together. What was the weight of the fish?

Suggestion.—Let x denote the weight of the body.

31. A person engaged a workman for 48 days. For each day that he labored he received 24 cents; and for each day that he was idle he paid 12 cents for his board. At the end of the 48 days the account was settled, when the laborer received 504 cents. Find how many days he worked, and how many days he was idle.

32. A fox, pursued by a grayhound, has a start of 60 leaps. He makes 9 leaps while the grayhound makes but 6; but 3 leaps of the grayhound are equivalent to 7 leaps

of the fox. How many leaps must the grayhound make to overtake the fox?

Suggestion.—Let x denote the number of leaps the grayhound must make before overtaking the fox. In the same time the fox will make $\frac{2}{3}x$ or $\frac{3}{2}x$ leaps.

$\frac{7}{3}x$ will denote the hound's whole distance.
 $\frac{3}{2}x$ “ “ “ fox's “ “

33. If we regard the minute hand of a clock as constantly pursuing and overtaking the hour hand, in what time does the minute hand gain in pursuit one of the 12 spaces of the dial?

Explanation.

Let x represent the time in minutes. Since in one hour, or 60 min., the minute hand gains 11 spaces, and 1 space in x min., we have the equation

$$11x = 60 \text{ min.}$$

$$\text{and } x = 5\frac{5}{11} \text{ min.}$$

34. If the hour and minute hands are now 6 spaces apart, how soon will they be together?

35. At what time between 9 and 10 o'clock will the minute and hour hands be together?

Suggestion.—At 9 o'clock the minute hand has 9 spaces to gain.

36. At what time between 7 and 8 o'clock will the hands of a clock point in exactly opposite directions?

Suggestion.—The required position holds the hands 6 spaces apart.

37. An estate of 9000 acres is to be divided among three persons, L., M., and N., so that M.'s share will be 387 acres less than L.'s, and N.'s share 2223 acres more than M.'s. Find the share of each.

38. An excursionist in going and returning has spent 8 hours. How far did he go if he went at the rate of 9 miles an hour, and came at the rate of 3 miles an hour?

39. A cistern can be filled by two pipes in 3 hours and 5 hours respectively. In what time can it be filled by the two pipes together?

40. The sum of $\frac{1}{7}$ and $\frac{1}{8}$ of a number is 150. Find the number.

41. Divide 49 into two such parts that, if the less be divided by 6 and the greater by 5, the sum of the quotients will be 9.

42. If a rectangular area is 6 yards longer and 4 yards shorter than the side of an adjacent square court of equal area, find the area of the court.

43. Find two numbers such that one may be n times as great as the other, and their sum equal to b .

44. A man agreed to work a days on the condition that for every day he worked he was to receive c cents, and for every day he was idle he was to forfeit d cents. At the end of a days he received m cents. How many days was he idle?

45. A man agreed to work 12 days on the condition that for every day he worked he was to receive 100 cents, and for every day he was idle he was to forfeit 50 cents. At the end of 12 days he received 900 cents. How many days was he idle?

NOTE.—Solve by the formula above, and also independently of the formula.

46. The fore wheel of a carriage is a feet, and the hind wheel is b feet in circumference. What is the distance passed over when the fore wheel has made c revolutions more than the hind wheel?

47. Two-thirds of A.'s money is equal to B.'s, and three-

fourths of B.'s money is equal to C.'s; together they have \$650. What amount has each?

48. A man having a hours for an excursion, rides away in a coach at the rate of b miles per hour, and returns on foot at the rate of c miles per hour. How far did he ride?

49. A. can plow a field in a days and B. in b days. In how many days can they plow the field working together?

50. A broker has two kinds of money: it takes n pieces of the first kind to make a dollar, and m pieces of the second kind; how many pieces of each kind must he take so that p pieces shall make a dollar?

Simple Equations containing Two Unknown Quantities.

The value of two unknown quantities cannot be obtained from a single equation, but from a *system* of equations. The equation $x + y = 12$ is not solvable; but by uniting with it a second equation, containing the same unknown quantities, as $x - y = 8$, solution may be made.

The process of solution combines the two equations so as to *eliminate* one of the two unknown quantities, and thus evolve a single equation containing but a single unknown quantity.

Three methods of elimination are in common use:

1. Elimination by Addition or Subtraction.
2. Elimination by Substitution.
3. Elimination by Comparison.

No method of elimination is available, however, unless the equations to be solved are *independent* and *consistent*.

$x + y = 12$ and $2x + 2y = 24$ are not independent equations, and $x + y = 5$ and $xy = 10$ are not consistent; therefore neither system can be solved.

Two equations, independent and consistent, are called *simultaneous* equations for the reason that the values found by solution satisfy both equations.

Elimination by Addition or Subtraction.

Solve the equations $\begin{cases} 7x + 2y = 31. & (1). \\ 3x - 4y = 23. & (2). \end{cases}$

Explanation.

Multiplying (1) by 2, $14x + 4y = 62.$ (3).

Writing (2), $\frac{3x - 4y = 23.}{} (2).$

Adding (2) and (3), $17x = 85.$

Hence, $\frac{x = 5.}{}$

Substituting this value in (2), $15 - 4y = 23.$

Hence, $4y = -8.$

$y = -2.$

EXERCISES.

Solve :

1. $\begin{cases} 2x - 3y = 4. \\ 6x - y = 28. \end{cases}$

2. $\begin{cases} x + 3y = 22. \\ 2x - 4y = 4. \end{cases}$

3. $\begin{cases} 2x + 3y = 7. \\ 5x + 7y = 19. \end{cases}$

4. $\begin{cases} 3x - y = 2. \\ 10x + 2y = 60. \end{cases}$

5. $\begin{cases} 3x + 10y = -1. \\ 2x + 7y = -\frac{1}{2}. \end{cases}$

6. $\begin{cases} 4x - 5y = 5. \\ 3x + 2y = 21. \end{cases}$

7. $\begin{cases} 10x - 3y = 25. \\ 5x - 9y = -25. \end{cases}$

8. $\begin{cases} 2x + 9y = -5. \\ 11x + 15y = 7. \end{cases}$

9. $\begin{cases} \frac{4x + 3y}{9} + y = 4. \\ x + \frac{8x - 2y}{5} = 7. \end{cases}$

10. $\begin{cases} \frac{4}{5 + y} = \frac{5}{12 + x}. \\ 2x + 5y = 35. \end{cases}$

Elimination by Substitution.

Solve the equations $\begin{cases} 2x - 5y = 1. & (1). \\ 7x + 3y = 24. & (2). \end{cases}$

Explanation.

Transposing $-5y$ in (1), and dividing by 2, we have $x = \frac{1+5y}{2}$.

Substituting this value in (2), we have $7 \times \frac{1+5y}{2} + 3y = 24$.

$$\text{Hence, } 7 + 35y + 6y = 48.$$

$$41y = 41.$$

$$y = 1.$$

Substituting this value in (1), we have $2x - 5 = 1$.

$$2x = 6.$$

$$x = 3.$$

EXERCISES.

Solve :

$$\begin{array}{l} 1. \quad x + 2y = 13. \\ \quad \quad 3x + y = 14. \end{array}$$

$$\begin{array}{l} 2. \quad 4x - y = 5. \\ \quad \quad 2x + 2y = 20. \end{array}$$

$$\begin{array}{l} 3. \quad 5x + 2y = 47. \\ \quad \quad 2x - y = 8. \end{array}$$

$$\begin{array}{l} 4. \quad \frac{1}{2}y - 3x = 2. \\ \quad \quad y = 14x. \end{array}$$

$$\begin{array}{l} 5. \quad 5x + 7y = 43. \\ \quad \quad 11x + 9y = 69. \end{array}$$

$$\begin{array}{l} 6. \quad 7x + 8y = 19. \\ \quad \quad 5x + 6y = 13\frac{1}{2}. \end{array}$$

$$\begin{array}{l} 7. \quad 5x - 3y = 17. \\ \quad \quad 4x + 2y = 16. \end{array}$$

$$\begin{array}{l} 8. \quad 7y - 21 = 5x. \\ \quad \quad 21x - 9y = 75. \end{array}$$

$$\begin{array}{l} 9. \quad 3x + 7y = 50. \\ \quad \quad 5x - 2y = 15. \end{array}$$

$$10. \quad x - \frac{3y}{5} = 6.$$

$$\frac{2x}{3} + 7y = 189.$$

Elimination by Comparison.

Solve the equations $\begin{cases} 6x + 5y = -8. & (1) \\ 4x + 3y = -5. & (2) \end{cases}$

Explanation.

Transposing $5y$ in (1) and dividing by 6, $x = \frac{-8 - 5y}{6}$. (3)

Transposing $3y$ in (2) and dividing by 4, $x = \frac{-5 - 3y}{4}$.

Equating these values of x , we have $\frac{-8 - 5y}{6} = \frac{-5 - 3y}{4}$.

Clearing of fractions, $-16 - 10y = -15 - 9y$.

Changing signs, $16 + 10y = 15 + 9y$.

Whence, $y = -1$

Substituting this value in (3), $x = \frac{-8 + 5}{6} = \frac{-3}{6} = -\frac{1}{2}$.

EXERCISES.

- | | |
|-----------------------|--|
| 1. $7x + 3y = 70.$ | 6. $2\frac{1}{2}x - 3\frac{1}{3}y = 10.$ |
| $5x - 4y = 7.$ | $7\frac{1}{3}x - 5\frac{1}{2}y = 55.$ |
| 2. $x + 8y = 17.$ | 7. $7x - 3y = 41.$ |
| $7x - 3y = 1.$ | $2x - y = 12.$ |
| 3. $11x - 2y = 5.$ | 8. $\frac{x}{4} + \frac{5y}{6} = 14.$ |
| $10x + 2y = 16.$ | $\frac{x}{8} + \frac{y}{6} = 4.$ |
| 4. $2x + 7y + 1 = 0.$ | 9. $14x - 3y = 39.$ |
| $5x + 8y - 7 = 0.$ | $6x + 17y = 35.$ |
| 5. $5x + 7y = 43.$ | 10. $ax + by = c.$ |
| $11x + 9y = 69.$ | $mx - ny = d.$ |

MISCELLANEOUS EXERCISES.

- | | |
|----------------------------------|-------------------------|
| 1. $x - y = 5.$ | 2. $x = 3y.$ |
| $\frac{x}{4} - \frac{y}{5} = 2.$ | $\frac{x}{3} + y = 34.$ |

$$3. \quad \begin{aligned} \frac{1}{2}x + \frac{1}{3}y &= 14. \\ \frac{1}{3}x + \frac{1}{2}y &= 11. \end{aligned}$$

$$4. \quad \begin{aligned} \frac{3x}{5} - \frac{2y}{7} &= 35. \\ x + 2y &= -63. \end{aligned}$$

$$5. \quad \frac{x}{3} + \frac{y}{4} = 3x - 7y - 37 = 0.$$

$$6. \quad \frac{x}{2} + \frac{y}{6} = 11.$$

$$7. \quad \frac{x+y}{2} = 2\frac{1}{2}.$$

$$\frac{x}{5} + \frac{y}{24} = \frac{7}{2}.$$

$$\frac{x-y}{3} = 3\frac{1}{3}.$$

$$\frac{x}{a} + \frac{y}{b} = m.$$

$$8. \quad \frac{x}{b} + \frac{y}{a} = n.$$

$$9. \quad \frac{x+1}{10} = \frac{3y-5}{2} = \frac{x-y}{8}.$$

$$10. \quad \frac{x+2y+1}{2x-y+1} = 2.$$

$$\frac{x+y}{x-y} = \frac{5}{3}.$$

$$\frac{3x-y+1}{x-y+3} = 5.$$

$$\frac{x+y+1}{x-y-1} = 7.$$

$$12. \quad x+ay = a(a+2b).$$

$$y - \frac{x}{b} = b.$$

$$13. \quad x+ay+a^2=0.$$

$$x+by+b^2=0.$$

$$14. \quad x = 4y.$$

$$\frac{1}{5}(2x+7y) - 1 = \frac{2}{3}(2x-6y+1).$$

$$15. \quad \frac{x}{5} - \frac{y}{4} = 0.$$

$$3x + \frac{1}{2}y = 17.$$

$$16. \quad (a+b)x + by = ax + (b+a)y = a^3 - b^3.$$

$$17. \quad \frac{x}{a+b} + \frac{y}{a-b} = 2a.$$

$$\frac{x-y}{4ab} = 1.$$

A Special Case.

The reciprocals of x and y are $\frac{1}{x}$ and $\frac{1}{y}$.

In certain fractional simultaneous equations x and y may be readily found by first finding the values of their reciprocals. If $\frac{1}{x} = a$, and $\frac{1}{y} = b$, then $x = \frac{1}{a}$, and $y = \frac{1}{b}$.

If $\frac{1}{x} = \frac{a}{b}$, then $x = \frac{b}{a}$, and so on.

$$1. \text{ Solve the equations } \frac{7}{x} + \frac{5}{y} = 19. \quad (1)$$

$$\frac{8}{x} - \frac{3}{y} = 7. \quad (2)$$

Explanation.

$$\text{Multiplying (1) by 3, } \frac{21}{y} + \frac{15}{y} = 57. \quad (3)$$

$$\text{Multiplying (2) by 5, } \frac{40}{x} - \frac{15}{y} = 35. \quad (4)$$

$$\text{Adding (3) and (4), } \frac{61}{x} = 92.$$

$$\text{Dividing by 61, } \frac{1}{x} = \frac{92}{61}.$$

$$\text{Hence } x = \frac{61}{92}.$$

In like manner, or by substitution, we find $y = \frac{61}{103}$.

Explanation.

$$2. \text{ Solve } \frac{a}{x} + \frac{b}{y} = m. \quad (1)$$

$$\text{Adding (1) and (2), } \frac{2a}{x} = m + n.$$

$$\frac{a}{x} - \frac{b}{y} = n. \quad (2)$$

$$\text{Dividing by } 2a, \quad \frac{1}{x} = \frac{m+n}{2a}$$

$$\text{Hence } x = \frac{2a}{m+n}.$$

EXERCISES.

$$1. \quad \frac{4}{x} + \frac{3}{y} = 2.$$

$$\frac{2}{x} - \frac{5}{y} = -7.$$

$$2. \quad \frac{5}{x} + \frac{6}{y} = 3.$$

$$\frac{15}{x} + \frac{3}{y} = 4.$$

$$3. \quad \frac{a}{x} + \frac{b}{y} = m.$$

$$\frac{c}{x} + \frac{d}{y} = n.$$

$$4. \quad \frac{5}{3x} + \frac{2}{5y} = 7.$$

$$\frac{7}{6x} - \frac{1}{10y} = 3.$$

$$5. \quad \frac{14}{x} + \frac{25}{y} = 7.$$

$$\frac{49}{x} - \frac{50}{y} = -3.$$

$$6. \quad \frac{5}{x} + \frac{13}{y} = 49.$$

$$\frac{7}{x} + \frac{3}{y} = 23.$$

$$7. \quad \frac{1}{x} + \frac{1}{y} = \frac{1}{n}.$$

$$\frac{1}{x} - \frac{1}{y} = n.$$

$$8. \quad \frac{9}{x} - \frac{5}{y} = \frac{2}{3}.$$

$$\frac{4}{x} + \frac{5}{y} = \frac{3}{2}.$$

$$9. \quad \frac{m}{nx} + \frac{n}{my} = m + n.$$

$$\frac{n}{x} + \frac{m}{y} = m^2 + n^2.$$

$$10. \quad \frac{m^2}{x} + \frac{n^2}{y} = mn(m + n).$$

$$\frac{n}{x} + \frac{m}{y} = m^2 + n^2.$$

Equations containing more than Two Unknown Quantities.

When more than two equations, independent and consistent, are to be solved, they must be reduced by elimination step by step, until there is obtained a single equation containing but a single unknown quantity. The process is as follows :

$$\text{Solve the equations } x + y + z = 53 \quad (1)$$

$$x + 2y + 3z = 107 \quad (2)$$

$$\underline{x + 3y + 4z = 137} \quad (3)$$

Explanation.

Subtracting (1) from (2), $y + 2z = 54$. (4)

Subtracting (1) from (3), $2y + 3z = 84$. (5)

We have now but two equations containing but two unknown quantities.

Multiplying (4) by 2, $2y + 4z = 108$. (6)

Writing (5), $2y + 3z = 84$. (5)

Subtracting (5) from (6), $z = 24$.

By substituting the value of z in (4) $y = 6$.

By substituting the values of y and z in (1) $x = 23$.

Solve the equations $2x + 3z = 21$. (1)

$$4y - z = 3. \quad (2)$$

$$\underline{3x + 2z = 19.} \quad (3)$$

Explanation.

Multiplying (2) by 3, $12y - 3z = 9$. (4)

Writing (1), $2x + 3z = 21$. (1)

Adding (1) and (4), $2x + 12y = 30$. (5)

Multiplying (2) by 2, $8y - 2z = 6$. (6)

Writing (3), $3x + 2z = 19$. (3)

Adding (3) and (6), $3x + 8y = 25$. (7)

We have now two equations, (5) and (7) containing but two unknown quantities, x and y .

Multiplying (5) by 3, $6x + 36y = 90$.

Multiplying (7) by 2, $6x + 16y = 50$.

Subtracting (7) from (5), $20y = 40$.

Whence, $y = 2$.

Substituting the value of y in (7), $x = 3$.

Substituting the value of x in (1), $z = 5$.

EXERCISES.

$$2x + 3y + 4z = 20. \quad x + y - z = 1.$$

$$1. \quad 3x + 4y + 5z = 26. \quad 3. \quad 8x + 3y - 6z = 1.$$

$$3x + 5y + 6z = 31. \quad 3z - 4x - y = 1.$$

$$3x - y - 2z = 11. \quad x + y + z = 26.$$

$$2. \quad 4x - 2y + z = -2. \quad 4. \quad x - y = 4.$$

$$6x - y + 3z = -3. \quad x - z = 6.$$

$$10x - y + 3z = 42.$$

$$5. \quad 7x + 2y + z = 51.$$

$$3x + 3y - z = 24.$$

$$x + y + z = 1.$$

$$6. \quad \frac{x}{2} + \frac{y}{4} + 4z = 1.$$

$$\frac{5}{3}x + \frac{3}{4}y - \frac{z}{2} = 1.$$

$$x - \frac{y}{5} = 6.$$

$$7. \quad y - \frac{z}{7} = 8.$$

$$z - \frac{x}{2} = 10.$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 36.$$

$$8. \quad \frac{1}{x} + \frac{3}{y} - \frac{1}{z} = 28.$$

$$\frac{1}{x} + \frac{1}{3y} + \frac{1}{2z} = 20.$$

$$\frac{1}{x} + \frac{1}{y} = 1.$$

$$9. \quad \frac{1}{x} + \frac{1}{z} = 2.$$

$$\frac{1}{y} + \frac{1}{z} = \frac{3}{2}.$$

$$\frac{x}{a} + \frac{y}{b} = 1.$$

$$10. \quad \frac{x}{a} + \frac{z}{c} = 1.$$

$$\frac{y}{b} + \frac{z}{c} = 1.$$

A Special Case.

$$\text{Solve } x + y + z = 6. \quad (1)$$

$$x + y + u = 7. \quad (2)$$

$$x + z + u = 8. \quad (3)$$

$$y + z + u = 9. \quad (4)$$

Explanation.

$$\text{Adding, } 3x + 3y + 3z + 3u = 30. \quad (5)$$

$$\text{Whence, } x + y + z + u = 10. \quad (6)$$

$$\text{Subtracting (1) from (6), } u = 4.$$

$$\text{" (2) " (6), } z = 3.$$

$$\text{" (3) " (6), } y = 2.$$

$$\text{" (5) " (6), } x = 1.$$

EXAMPLES.

Solve:

$$y + z = 2.$$

$$x + y = 26.$$

$$1. \quad z + u = -1.$$

$$y + z = 41.$$

$$u + y = 3.$$

$$4. \quad z + w = 34.$$

$$2x + y + z = 16.$$

$$w + u = 15.$$

$$2. \quad x + 2y + z = 9.$$

$$u + x = 16.$$

$$x + y + 2z = 3.$$

$$-x + y + z + u = a.$$

$$x + y - 8 = 0.$$

$$5. \quad x - y + z + u = b.$$

$$3. \quad y + z - 28 = 0.$$

$$x + y - z + u = c.$$

$$x + z - 14 = 0.$$

$$x + y + z - u = d.$$

PROBLEMS.

Solved by two or more Independent Equations.

Problems, to be solved by means of simultaneous equations, must supply as many independent conditions as there are quantities to be found.

1. The sum of two numbers is 25, and their difference is 7. Find the numbers.

Explanation.

Letting x and y represent the numbers,

we have

$$x + y = 25. \quad (1) \text{ by the first condition.}$$

and

$$x - y = 7. \quad (2) \text{ by the second condition.}$$

Adding (1) and (2)

$$2x = 32.$$

Whence,

$$x = 16, \text{ the greater.}$$

Subtracting (2) from (1),

$$2y = 18.$$

Whence,

$$y = 9, \text{ the less.}$$

2. If the numerator of a certain fraction is increased by 1, the value of the fraction becomes 1; but if the numerator is diminished by 1, the value of the fraction becomes $\frac{3}{4}$. Find the fraction.

Explanation.

Letting x and y represent the numerator and denominator,

we have $\frac{x+1}{y} = 1$. (1) by the first condition.

and $\frac{x-1}{y} = \frac{3}{2}$. (2) by the second condition.

From (1), $x - y = -1$. (3)

From (2), $4x - 3y = 4$. (4)

Multiplying (3) by 3, $3x - 3y = -3$. (5)

Writing (4), $4x - 3y = 4$. (4)

Subtracting (5) from (4), $x = 7$, the numerator.

Substituting in (3), $7 - y = -1$.

Whence $y = 8$, the denominator.

Hence, the fraction required is $\frac{7}{8}$.

3. The middle digit of a number consisting of three Arabic figures is 0; the sum of the other digits is 10, and if their order be reversed, they then express a number greater than the original number by 396. Find the original number.

Explanation.

Letting x represent the right-hand digit, and y the left-hand digit, we have $100x + y$ to represent the units in the number.

Reversing the position of x and y , we have $100y + x$ to represent the second number formed.

Hence, we have the equation, $100y + x - (100x + y) = 396$. (1)

Since the sum of the other digits is 10, we have $x + y = 10$. (2)

Whence $x = 3$, and $y = 7$. Hence the number sought is 307.

1. One-nineteenth of the sum of two numbers is 14, and one-half their difference is 4; find the numbers.

2. If 2 is subtracted from both numerator and denominator of a certain fraction, its value is $\frac{2}{3}$; and if 1 is added to both numerator and denominator, its value is $\frac{2}{3}$. What is the fraction?

3. A. has £5 and B. has 10 shillings. How much must

A. give to B. in order that he may have just four times as much as B. ?

4. Divide 50 into two parts, such that twice one part is equal to three times the other.

5. A. has \$5 less than B., C. has as much as A. and B. together, and A., B., and C. have \$50 between them. How much has each ?

6. One man is 70 and another is 45 years of age. When was the first twice as old as the second ?

7. What fraction is that, to whose numerator if 1 be added, its value will be $\frac{1}{3}$; but if 1 be added to the denominator its value will be $\frac{1}{4}$?

8. Find two numbers whose sum is a , and whose difference is b .

9. The sum of two digits of a number is 9, and if 27 is subtracted from the number, the digits will be reversed. Find the number.

10. Eight years ago, A. was four times as old as B. ; but in 12 years he will be only twice as old. Required their present ages.

11. A. says to B., "Give me \$100 of your money, and I shall have as much money as you." B. replies, "Give me \$100 of your money, and I shall have twice as much as you." How many dollars has each ?

12. A. and B. can do a piece of work in 12 days, B. and C. in 16 days, and A. and C. in 20 days. How long will it take each to do it ?

13. A man sells at one time 5 sheep and 10 calves for \$90, and at another time, at the same rate, 10 sheep and 5 calves for \$75. What was the selling price of a sheep and of a calf ?

14. One girl said to another, "Give me 5 of your roses, and I shall have three times as many as you will have left. Said the other, "Give me 2 of your roses, and I shall have five times as many as you will have left." How many roses had each girl?

15. A boatman rows 7 miles with the stream in 1 hour, and returns against the stream in $2\frac{1}{2}$ hours. At what rate would he row in still water, and at what rate does the stream flow?

16. A trader allows \$500 for expenses, and augments that part of his capital which is not so expended by $\frac{1}{3}$ of it; at the end of three years his stock is doubled. What had he at first?

17. A company of infantry are 1200 of their own paces in advance of a troop of cavalry. The former take 5 paces to every 4 of the latter, but 3 paces of the latter equal in extent 4 paces of the former. How far in paces will the cavalry march before they overtake the infantry?

18. A rectangle has the same area as a second rectangle, but is 4 yards shorter and two yards wider; it has also the same area as a third rectangle, but is 3 yards shorter and $1\frac{3}{5}$ yards wider. What are its dimensions?

19. A., B., and C. can do a piece of work in 10 days; B., C., and D., in 12 days; A., C., and D., in 15 days; and A., B., and D., in 18 days. How long will it take each to do it?

20. A person who possessed \$100,000 placed the greater part of it out at 5% interest, and the other part at 4%. The whole income received was \$4640. Required the two parts.

THREE UNKNOWN QUANTITIES.

1. A., B., C., and D. have \$270 among them; A. has three times as much as C., and B. five times as much as D.; also A. and B. together have \$50 less than eight times what C. has. Find how much each has.

A. B. C. D.

Suggestion.— $x + y + z + u = 270$.

$$x = 3z. \quad y = 5u.$$

$$x + y = 8z - 50.$$

2. \$1000 is divided among E., F., G., and H. F. gets half as much as E.; G.'s share exceeds H.'s by as much as one-third of E.'s; and if F.'s were increased by \$100, he would have as much as both G. and H. How much does each get?

3. A number consists of 3 digits whose sum is 10. The middle digit is equal to the sum of the other two; and the number will be increased by 99 if its digits be reversed. Find the number.

4. A man has three ingots composed of three different metals in different proportions. A pound of the first consists of 7 ounces of silver, 3 of copper, 6 of tin; one pound of the second consists of 12 ounces of silver, 3 of copper, 1 of tin; a pound of the third consists of 4 ounces of silver, 7 of copper, 5 of tin. How much of each of the ingots must be taken to form another of one pound weight consisting of 8 ounces of silver, $3\frac{3}{4}$ of copper, $4\frac{1}{4}$ of tin?

5. Three men, A., B., and C., had together a certain sum of money. Now, if A. gives to B. and C. as much as they already have, and then B. gives to A. and C. as much as they have after the first distribution, and again C. gives to

A. and B. as much as they have after the second distribution, they will each have \$6. How much did each have at first?

Generalization.

By generalization in algebra, formulæ are obtainable, which, owing to their applicability to all problems of the same class, greatly facilitate the process of solution.

1. The sum of two numbers is a , and their difference is b . Find the two numbers.

Explanation.

Letting x represent the larger number, and y the smaller number, we have the equations:

$$1. \ x + y = a.$$

$$2. \ x - y = b.$$

Solving, we have $x = \frac{a+b}{2}$, the greater number,

and $y = \frac{a-b}{2}$, the smaller number.

These results are formulæ which may be used in the solution of like problems.

Given the sum = 10, the difference = 6, the numbers are

$$\frac{10+6}{2} \text{ and } \frac{10-6}{2}; \text{ that is, 8 and 2.}$$

Given the sum = 7, the difference = 6, the numbers are

$$\frac{7+6}{2} \text{ and } \frac{7-6}{2}; \text{ that is, } 6\frac{1}{2} \text{ and } \frac{1}{2}.$$

2. What is the interest of p dollars for y years at r per cent.?

Explanation.

Letting x represent the interest, we have

$$1. \ x = p \times \frac{r}{100} \times y = \frac{pry}{100}.$$

Since $\frac{\text{months}}{12} = y$, we write $\frac{m}{12} = y$.

Substituting this for y in (1), we have

2. $x = \frac{prm}{1200}$, the formula to be used when the time is expressed in months.

Since $\frac{\text{days}}{360} = y$, we write $\frac{d}{360} = y$.

Substituting this for y in (1), we have

3. $x = \frac{prd}{36000}$, the formula to be used when the time is expressed in days.

To apply these formulæ: What is the interest of \$600 for 6 yr. 6 mos. 6 da. at 6%?

Explanation.

$$6y \ 6m \ 6d = 6\frac{3\frac{1}{6}}{6}y.$$

$$1. \ x = \frac{pry}{100} = \frac{600 \times 6 \times 6\frac{3\frac{1}{6}}{6}}{100} = \$234.60.$$

$$6y \ 6m \ 6d = 78.2m.$$

$$2. \ x = \frac{prm}{1200} = \frac{600 \times 6 \times 78.2}{1200} = \frac{469.2}{2} = \$234.60.$$

$$6y \ 6m \ 6d = 2346d.$$

$$3. \ x = \frac{prd}{36000} = \frac{600 \times 6 \times 2346}{36000} = \frac{2346}{10} = \$234.60.$$

Specific Gravity.

3. The specific gravity of a body is the ratio between the weight of the body and the weight of an equal volume of water.

Explanation.

If w is the weight of a solid in air and w^1 is its weight in water, then $w - w^1$ = the weight of an equal volume of water, and we have

$$\text{sp. gr.} = \frac{w}{w - w^1}.$$

A body in air weighs $18\frac{1}{8}$ ounces, and in water 15 ounces; what is its specific gravity?

Explanation.

$$\text{Sp. gr.} = \frac{w}{w - w^1} = \frac{18\frac{1}{8}}{18\frac{1}{8} - 15} = \frac{18\frac{1}{8}}{3\frac{1}{8}} = 5.8.$$

Thermometers.

4. Thermometers are of three principal kinds : The *Fahrenheit*, the *Centigrade*, and the *Réaumur*.

It is sometimes necessary to transform readings from one scale to another.

	Freezing-point.	Boiling-point.	Inter-space.
Fahrenheit	32°	212°	180°
Centigrade	0°	100°	100°
Réaumur	0°	80°	80°

Hence the number of degrees F. $- 32^\circ = \frac{180}{160}$ or $\frac{9}{8}$ C. $= \frac{180}{80}$ or $\frac{9}{4}$ R.

From which we deduce the following formula :

- | | |
|--|---|
| 1. $(F. - 32^\circ) \times \frac{5}{9} = C.$ | 3. $C. \times \frac{9}{5} = F. - 32^\circ.$ |
| 2. $(F. - 32^\circ) \times \frac{4}{9} = R.$ | 4. $R. \times \frac{9}{4} = F. - 32^\circ.$ |

Change 70° F. to C. and R.

Explanation.

$$(F. - 32^\circ) \times \frac{5}{9} = C. \text{ and } (F. - 32^\circ) \times \frac{4}{9} = R.$$

$$(70^\circ - 32^\circ) \times \frac{5}{9} = 38^\circ \times \frac{5}{9} = \frac{190}{9}^\circ = 21\frac{1}{9}^\circ \text{ C.}$$

$$(70^\circ - 32^\circ) \times \frac{4}{9} = 38^\circ \times \frac{4}{9} = \frac{152}{9}^\circ = 16\frac{8}{9}^\circ \text{ R.}$$

1. Change 32° F. to C. and R.
2. Change $- 10^\circ$ F. to C. and R.

Work.

A. can do a piece of work in a days, B. in b days, and C. in c days. In what time can they all do the work together?

Explanation.

Letting x represent the time required, we have $\frac{x}{a} + \frac{x}{b} + \frac{x}{c} = 1.$

Solving, we have $bcx + acx + abx = abc$, and $x = \frac{abc}{bc + ac + ab},$
the days required.

This formula will solve any like numerical problem.

INVOLUTION.

Involution treats of the production of powers ; and powers, as we have seen, result from the répétitions of a certain quantity as a factor, the number of répétitions being indicated by the exponent of the power.

The 7th power of $a = a \times a \times a \times a \times a \times a \times a = a^7$, the exponent 7 indicating the seven répétitions of a .

$$\left. \begin{aligned} (+a)^2 &= (+a) \times (+a) = +a^2 \\ (-a)^2 &= (-a) \times (-a) = +a^2 \end{aligned} \right\} \text{even powers.}$$

$$\left. \begin{aligned} (+a)^3 &= (+a) \times (+a) \times (+a) = +a^3 \\ (-a)^3 &= (-a) \times (-a) \times (-a) = -a^3 \end{aligned} \right\} \text{odd powers.}$$

From which it is obvious that

1. *An even power of either a positive or a negative quantity is positive.*
2. *An odd power of a positive quantity is positive.*
3. *An odd power of a negative quantity is negative.*

Involution of Monomials.

1. Involve $(3a^2b)^4$ as indicated.

Explanation.

$$(3a^2b)^4 = (3a^2b) (3a^2b) (3a^2b) (3a^2b) = 3^4(a^2)^4b^4 = 81a^8b^4$$

2. Involve $(-5mn^2)^3$ as indicated.

Explanation.

$$(-5mn^2)^3 = (-5mn^2) (-5mn^2) (-5mn^2) = (-5)^3m^3(n^2)^3 = -125m^3n^6.$$

Hence brief directions are :

1. Give to each factor of the monomial the exponent of the required power.
2. Involve each factor as this exponent indicates.

4. Prefix — to the result when, at the same time, the monomial is negative and the power odd.

1. Raise $(2a^2b^3)^5$ to the power required.

$$2^5 (a^2)^5 (b^3)^5 = 32a^{10}b^{15}. \text{ Ans.}$$

2. Raise $(-2a^2b^3)^5$ to the power required.

$$-2^5 (a^2)^5 (b^3)^5 = -32a^{10}b^{15}. \text{ Ans.}$$

EXERCISES.

Involve:

1. $(a^4c^2)^3$.

6. $(3a^3b^4c)^6$.

2. $(3a^2b^3)^4$.

7. $(-6xy^7z)^3$.

3. $(-5ab^2c^3)^4$.

8. $(3m^mb^c)^5$.

4. $(-5ab^2c^3)^5$.

9. $(+4ab^2c)^m$.

5. $(-7x^2yz^2)^3$.

10. $(+9x^6y^3z^{15})^n$.

Involution of Fractions.

A fraction is raised to any given power by raising both numerator and denominator to that power.

$$\left(-\frac{3a^5}{5x^3}\right)^2 = \frac{9a^{10}}{25x^6}.$$

1. $\left(\frac{1}{2a^2}\right)^5$.

6. $\left(-\frac{2x^5}{3a^4}\right)^6$.

2. $\left(\frac{3x^4}{2x^3}\right)^5$.

7. $\left(-\frac{a^2b^3}{cd^4}\right)^4$.

3. $\left(\frac{2x^3}{3y}\right)^8$.

8. $\left(\frac{2a^2b^3c}{3xy^2z^4}\right)^m$.

4. $\left(-\frac{x^3}{3}\right)^7$.

9. $\left(-\frac{a^{-m}b^2}{x^my^n}\right)^5$.

5. $\left(\frac{1}{2a^2}\right)^7$.

10. $\left\{\left(\frac{a^3b^2}{c^2}\right)^2\right\}^3$.

Involution of Binomials.

By actual multiplication we obtain the following results :

Binomials. Factors. Expansion.

$$(a + b)^2 = (a + b)(a + b) = a^2 + 2ab + b^2.$$

$$(a - b)^2 = (a - b)(a - b) = a^2 - 2ab + b^2.$$

$$(a + b)^3 = (a + b)(a + b)(a + b) = a^3 + 3a^2b + 3ab^2 + b^3.$$

$$(a - b)^3 = (a - b)(a - b)(a - b) = a^3 - 3a^2b + 3ab^2 - b^3.$$

In like manner

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

$$(a - b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4.$$

$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

$$(a + b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6.$$

$$(a - b)^6 = a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6.$$

The obvious laws of expansion are :

1. *The exponent of the binomial + 1 = the number of terms.*

2. *When b is positive the signs of all the terms are positive ; when b is negative the signs of the terms are alternately positive and negative.*

3. *The exponents of a and b stand arranged, a starting in the first term with the exponent of the binomial, and b in the second term with the exponent 1.*

4. *The coefficient of the first term is 1 ; that of the second term is the exponent of the binomial.*

5. *Any term supplies the coefficient of the next term as follows :*

1. *Multiply the coefficient by the exponent of a.*

2. *Divide the result by the exponent of b plus 1.*

Expand the following without actual multiplication :

- | | |
|------------------|-------------------|
| 1. $(x + y)^2$. | 6. $(m + n)^8$. |
| 2. $(x + 1)^2$. | 7. $(m - n)^9$. |
| 3. $(x - y)^3$. | 8. $(a - 3)^3$. |
| 4. $(1 + a)^4$. | 9. $(3 - x)^4$. |
| 5. $(x - 4)^5$. | 10. $(y - 2)^6$. |

Expand $(2p - q^2)^5$.

Explanation.

Letting x represent $2p$, and y represent q^2 , we have $(x - y)^5 = x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5$.

Substituting $2p$ for x , and q^2 for y , we have $(2p)^5 - 5(2p)^4q^2 + 10(2p)^3(q^2)^2 - 10(2p)^2(q^2)^3 + 5(2p)(q^2)^4 - (q^2)^5$.

Involving we have $32p^5 - 80p^4q^2 + 80p^3q^4 - 40p^2q^6 + 10pq^8 - q^{10}$.

Expand the following :

- | | |
|----------------------|--------------------|
| 1. $(a - 2x)^3$. | 4. $(2m + 3n)^5$. |
| 2. $(4a + 3b)^4$. | 5. $(3p - 2q)^7$. |
| 3. $(x^2 + y^2)^6$. | 6. $(100 - 2)^2$. |

Involution of a Polynomial.

Find the square of $a + b + c$.

$$\begin{array}{r}
 a + b + c \\
 a + b + c \\
 \hline
 a^2 + ab + ac \\
 \quad ab + b^2 + bc \\
 \quad \quad ac + bc + c^2 \\
 \hline
 a^2 + 2ab + 2ac + b^2 + 2bc + c^2 = \\
 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.
 \end{array}$$

In like manner, we find

$$\begin{aligned}
 (a + b + c + d)^2 &= a^2 + b^2 + c^2 + d^2 + 2ab + 2ac + \\
 &\quad 2ad + 2bc + 2bd + 2cd.
 \end{aligned}$$

Hence we have the

RULE.

1. Find the square of each term.
2. Find twice the product of each term and each of the following terms.
3. Connect the results with their proper signs.

Square $2a + b + c - d$.

$$(2a + b + c - d)^2 = 4a^2 + b^2 + c^2 + d^2 + 4ab + 4ac - 4ad + 2bc - 2bd - 2cd.$$

Expand the following :

- | | |
|------------------------|---------------------------------|
| 1. $(a + b + c)^2$. | 6. $(a + b + c + d)^2$. |
| 2. $(x + y - z)^2$. | 7. $(a - b - c + d)^2$. |
| 3. $(a - b - c)^2$. | 8. $(x - y + z - u)^2$. |
| 4. $(l - m + p)^2$. | 9. $(1 + m + m^2 + m^3)$. |
| 5. $(1 + x + x^2)^2$. | 10. $(2x^3 - 3x^2 - x + 3)^2$. |

Cube $m + n + o$. $(m + n + o)^3 = \overline{(m + n + o)^3} =$
 $(m + n)^3 + 3(m + n)^2o + 3(m + n)o^2 + o^3 =$
 $m^3 + 3m^2n + 3mn^2 + n^3 + 3m^2o + 6mno + 3n^2o + 3mo^2$
 $+ 3no^2 + o^3$
 $= \underline{m^3 + n^3 + o^3} + \underline{3m^2n + 3m^2o + 3n^2o + 3mn^2 + 3mo^2}$
 $+ \underline{3no^2 + 6mno}.$

Hence the cube of a trinomial, whose terms are positive, consists of :

1. *The cube of each of the three terms, plus*
2. *Three times the square of each term multiplied by each of the other terms, plus*
3. *Six times the product of the three terms.*

Expand $(a + b - c)^3$.

$$(a + b - c)^3 = a^3 + b^3 - c^3 + 3a^2b - 3a^2c + 3ab^2 - 3b^2c + 3ac^2 + 3bc^2 - 6abc.$$

Why are the 3d, 5th, 7th, and 10th terms negative?

Expand :

1. $(a - b + c)^3$.

4. $(2 - 3x + 4x^2)^3$.

2. $(a - b - c)^3$.

5. $(m^2 + n^2 - o^2)^3$.

3. $(1 - a + a^2)^3$.

6. $(x^2 + 2x - 3)^3$.

EVOLUTION.

Evolution treats of the extraction of roots ; and a root, as we have seen, is one of the equal factors of a quantity. Evolution is the inverse of Involution. Since the square of $a = aa = a^2$, the square root of $a^2 = a$; and since the cube of $a = aaa = a^3$, the cube root of $a^3 = a$.

$\sqrt{}$, the Radical Sign, prefixed to a quantity, indicates that a root is to be taken.

$\sqrt{a}$ indicates the square root of a .

$\sqrt[3]{a}$ indicates the cube root of a .

$\sqrt[4]{a}$ indicates the fourth root of a .

The figure written in the angle of the sign, as 3 or 4, is called the index of the root. A sign without an index denotes the square root.

Evolution of Monomials.

1. $(a^3)^2 = a^6$ (2 being multiplier).

$\therefore \sqrt{a^6} = a^3$ (the index 2, understood, being divisor).

2. $(a^4)^3 = a^{12}$ (3 being multiplier).

$\therefore \sqrt[3]{a^{12}} = a^4$ (the index 3 being divisor).

RULE.

Divide the exponents of the monomial by the index of the required root.

EXERCISES.

1. Find the cube root of
- x^3y^9
- , and of
- $-x^3y^9$
- .

Process.

$$\sqrt[3]{x^3y^9} = xy^3.$$

$$\sqrt[3]{-x^3y^9} = -xy^3.$$

2. Find the 3d root of
- $-27a^6b^3$
- .

Process.

$$\sqrt[3]{-27a^6b^3} = \sqrt[3]{-3^3a^6b^3} = -3a^2b.$$

3. Find the 4th root of
- $81m^8n^4$
- , and of
- $-81m^8n^4$
- .

Process.

$$\sqrt[4]{81m^8n^4} = \sqrt[4]{3^4m^8n^4} = \pm 3m^2n.$$

$$\sqrt[4]{-81m^8n^4} = \sqrt[4]{-3^4m^8n^4} = \text{an impossible root,}$$

since $(-3)^4 = +81$.

$\therefore$ The even root of a negative quantity is impossible.

Since $(+a) \times (+a) = a^2$, the square root of $a^2 = +a$ or $-a$.
and $(-a) \times (-a) = a^2$.

The 4th root of $a^4 = \pm a$.

$$\sqrt[5]{x^5} = x, \text{ but not } -x; \text{ why not?}$$

$$\sqrt[5]{-x^5} = -x, \text{ but not } +x; \text{ why not?}$$

RULE.

Give every even root of a positive quantity the sign $\pm$,
and every odd root of a quantity the sign of the quantity.

Find the required roots of the following :

1. $\sqrt{36a^6b^4c^2}.$

6. $\sqrt{16a^4b^8}.$

2. $\sqrt[3]{-8x^3y^6z^3}.$

7. $\sqrt[4]{81m^{16}n^{16}}.$

3. $\sqrt{81q^4r^8s^{10}}.$

8. $\sqrt[5]{-32a^{5n}b^{10m}}.$

4. $\sqrt{49l^6m^8n^{12}}.$

9. $\sqrt[6]{64x^{24}y^{18}z^{12}}.$

5. $\sqrt[3]{-125m^3n^6x^{12}}.$

10. $\sqrt[3]{x^{3n} + 3y^{12m} - 9}.$

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}; \therefore \sqrt[n]{\frac{a^n}{b^n}} = \frac{\sqrt[n]{a^n}}{\sqrt[n]{b^n}} = \frac{a}{b}.$$

PRINCIPLE.

The root of a fraction is a fraction composed of the required roots of its terms.

$$1. \sqrt[4]{\frac{16m^2n^4}{25m^8}}$$

$$4. \sqrt[4]{\frac{a^4}{81b^8c^4}}$$

$$2. \sqrt[3]{\frac{125a^3b^6c^9}{2a^6b^9}}$$

$$5. \sqrt[5]{-\frac{a^{5m}}{243y^{10}}}$$

$$3. \sqrt[3]{-\frac{l^{15}m^{12}}{27x^9}}$$

$$6. \sqrt[n]{\sqrt[m]{\frac{x^{3mn}y^{2mn}}{a^{mn}b^{mn3}}}}$$

Square Root of Polynomials.

Since $(a + b)^2 = a^2 + 2ab + b^2$, the square root of $a^2 + 2ab + b^2 = a + b$.

It is important, however, to ascertain by what steps the root $a + b$ may be found by evolution.

Process.

$$\begin{array}{r} a^2 + 2ab + b^2(a + b \\ a^2 \\ \hline 2a + b \mid 2ab + b^2 \\ 2ab + b^2 \\ \hline 0 \end{array}$$

b to $2a$, making $2a + b$, and this result multiplied by b , and the product subtracted, leaves no remainder.

Explanation.

$\sqrt{a^2} = a$, the first term of the root. Subtracting a^2 , we have $2ab + b^2$.

$2ab \div 2a = b$, the second term of the root.

To complete the divisor we add

Brief directions are :

1. Arrange What ?
2. Extract root Of what ?
3. Divide by twice the root Divide what ?
4. (1) Annex and (2) Multiply. 1. What ? 2. By what ?
5. Subtract What ?
6. Repeat 3, 4, and 5 when necessary.

EXERCISES.

Extract the square root of :

1. $a^4 - 4a^3 + 8a + 4$.
2. $4x^4 + 12x^3 + 5x^2 - 6x + 1$.
3. $4x^2 - 12xy + 9y^2$.
4. $x^4 - 4x^3 + 2x^2 + 4x + 1$.
5. $16x^4 + 24x^3 + 89x^2 + 60x + 100$.
6. $a^2 - ab + \frac{1}{4}b^2$.
7. $4a^4 + 4a^3 - 7a^2 - 4a + 4$.
8. $1 + 4x + 10x^2 + 12x^3 + 9x^4$.
9. $4x^4 + 4x^3 - \frac{1}{2}x + \frac{1}{16}$.
10. $4m^2 + \frac{9}{m^2} - \frac{6}{m} - 11 + 4m$.
11. $1 + x$.

Process.

$$\begin{array}{r}
 1 + x \overline{) 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} \dots\dots\dots} \\
 \underline{1} \phantom{+ \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} \dots\dots\dots} \\
 2 + \frac{x}{2} \overline{) + x} \\
 \phantom{2 + \frac{x}{2} \overline{) }} \underline{+ x + \frac{x^2}{4}} \\
 2 + x - \frac{x^2}{8} \overline{) - \frac{x^2}{4}} \\
 \phantom{2 + x - \frac{x^2}{8} \overline{) }} \underline{- \frac{x^2}{4} - \frac{x^3}{8} + \frac{x^4}{64}} \\
 2 + x - \frac{x^2}{4} + \frac{x^3}{16} \overline{) \frac{x^3}{8} - \frac{x^4}{64}, \text{Remainder.}}
 \end{array}$$

SQUARE ROOT OF NUMBERS.

Periods and Roots Compared.

Separating the following squares into two-figure periods as far as possible, we have :

$$\begin{array}{ccc} 1' & 10' & 1'00'00. \\ 81' & 98'01 & 99'80'01. \end{array}$$

Using the radical sign we have :

One period.	Two periods.	Three periods.
$\sqrt{1'} = 1.$	$\sqrt{1'00} = 10.$	$\sqrt{1'00'00} = 100.$
$\sqrt{81'} = 9.$	$\sqrt{98'01} = 99.$	$\sqrt{99'80'01} = 999.$

Obviously, one period in the square gives but one figure in the root ; two periods in the square, two figures in the root ; three periods in the square, three figures in the root. Hence the principle :

The number of figures in the square root equals the number of two-figure periods into which the square can be pointed off, beginning at units.

NOTE.—The period on the extreme left may contain but a single figure.

Letting t represent the tens of a root and u the units, we have the formula $(t + u)^2 = t^2 + 2tu + u^2$.

By means of the above principle and formula the extraction of the square root of numbers is readily explained.

Extract the square root of 3249.

Explanation.

Pointing off we have	Square.	$t \ u \ \text{Root.}$
Greatest square in 32 =	$32'49 = t^2 + 2tu + u^2$	$\begin{array}{l} 5 \ 7 \end{array}$
	$\begin{array}{r} 25 \\ \hline 749 \end{array}$	$\begin{array}{r} = t^2 \\ \hline = 2tu + u^2 \end{array}$
$2tu \div 2t = u$		$= (2t + u) u$
$\therefore 70 \div 2 \times 5 \text{ or } 10 = 7 \text{ units.}$		
$(2t + u) = 10 \text{ tens} + 7 \text{ units} = 107.$		
$(2t + u) u = 107 \times 7 =$	$\begin{array}{r} 749 \\ \hline 0 \end{array}$	$= (2t + u) u.$

Hence, brief directions are :

1. Point off the number into two-figure periods.
2. Find, in the first period, the greatest square and its root.

3. Subtract, and annex the next period for a remainder.

4. Divide the remainder by twice the root.

5. Annex the quotient to both root and divisor.

6. Multiply the complete divisor by the quotient.

7. Apply (3), (4), (5), and (6) again, if necessary.

To apply the rule :

Find the square root of 103041.

Explanation.

		root. quotient. quotient. (5)
Pointing off	10'30'41(3 2 1	
Greatest square	<u>9</u>	
Twice the root, 6	130 remainder	
With quotient annexed 62	<u>124</u> product of common divisor and quotient	
Applying (3) (3)	<u>641</u>	
Applying (4) 64 (6)	<u>641</u>	
Applying (5) 641	0	

EXERCISES.

Find the square root of :

1. 5329.

6. 170569.

2. 53824.

7. 582169.

3. 616225.

8. 12574116.

4. 141376.

9. 29735209.

5. 160801.

10. 3661097049.

Square Root of Common and Decimal Fractions.

1. What is the square root of $\frac{1}{4}$?

$$\sqrt{\frac{1}{4}} = \frac{1}{2}; \text{ for } \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}.$$

2. What is the square root of $\frac{4}{9}$?

$$\sqrt{\frac{4}{9}} = \frac{2}{3}; \text{ for } \frac{2}{3} \times \frac{2}{3} = \frac{4}{9}. \text{ Hence, } \sqrt{\text{Fraction}} = \frac{\sqrt{\text{Numerator}}}{\sqrt{\text{Denominator}}}.$$

3. Find the square root of .25.

$$\sqrt{.25} = .5; \text{ for } .5 \times .5 = .25.$$

4. The $\sqrt{.4}$ = what?

$$\text{Not } .2, \text{ for } .2 \times .2 = .04.$$

$$\text{But } \sqrt{.4} = \sqrt{.40} = \sqrt{.4000} = .632 +.$$

In such cases annex ciphers and make periods *from the decimal point toward the right*.

1. Find the square root of:

1. $\frac{1}{9}$.	5. $\frac{81}{169}$.	9. $\frac{2304}{5184}$.	13. $\frac{898704}{935089}$.
2. $\frac{4}{16}$.	6. $\frac{121}{441}$.	10. $\frac{2704}{4225}$.	14. $\frac{1000000}{100000000}$.
3. $\frac{9}{100}$.	7. $\frac{64}{225}$.	11. $\frac{55225}{784996}$.	15. $\frac{1002001}{1234321}$.
4. $\frac{100}{10000}$.	8. $\frac{225}{729}$.	12. $\frac{781456}{978121}$.	16. $\frac{4937284}{1110889}$.

2. Find the square root of:

1. .09.	6. .12345.
2. .9.	7. .763876.
3. .0144.	8. .30858025.
4. .144.	9. .00009801.
5. .0100.	10. .00010201.

Find the square root of:

1. $\frac{2}{3}$.

Process.

$$\sqrt{\frac{2}{3}} = \sqrt{\frac{6}{9}} = \frac{\sqrt{6}}{3} = \frac{2.45}{3} = .81\frac{2}{3}. \text{ Ans.}$$

2. $\frac{7}{8}$.

4. $\frac{144}{1728}$.

3. $\frac{8}{9}$.

5. $\frac{299}{2704}$.

6. $\frac{11}{16}$.

8. $\frac{1}{2}$.

7. $\frac{12}{25}$.

9. $\frac{3}{4}$.

10. $\frac{3}{4} + \frac{5}{5} + \frac{6}{9}$.

Cube Root of Polynomials.

Since $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, the cube root of $a^3 + 3a^2b + 3ab^2 + b^3 = a + b$.

It is important, however, to ascertain by what steps the root $a + b$ may be found by evolution.

Process.

$$\begin{array}{r}
 a^3 + 3a^2b + 3ab^2 + b^3 \mid a + b \\
 \underline{a^3} \\
 3a^2 \mid 3a^2b + 3ab^2 + b^3 = (3a^2 + 3ab + b^2) b \\
 3a^2 + 3ab + b^2 \mid 3a^2b + 3ab^2 + b^3 \\
 \underline{\hspace{1.5cm}} \\
 0
 \end{array}$$

Explanation.

$\sqrt[3]{a^3} = a$, the first term of the root. Subtracting a^3 , we have $3a^2b + 3ab^2 + b^3$.

$3a^2b \div 3a^2 = b$, the second term of the root. The remainder factored $= (3a^2 + 3ab + b^2) b$; hence, to complete the divisor $3a^2$, we must add to it $3ab + b^2$, making $3a^2 + 3ab + b^2$; this result multiplied by b and the product subtracted leaves no remainder.

Brief directions are :

1. Arrange What ?
2. Extract root Of what ?
3. Divide by 3 times the square of the root. Divide what ?
4. Annex, and annex again What ?
5. Multiply By what ?
6. Subtract What ?
7. Repeat 3, 4, 5, and 6 when necessary.

EXERCISES.

Find the cube root of :

Process.

$$\begin{array}{r}
 1. \quad x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1 \quad (x^2 - 2x + 1 \\
 \quad \underline{x^6} \\
 \quad \quad - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1 \\
 \quad \quad \underline{- 6x^5 + 12x^4 - 8x^3} \\
 \quad \quad \quad 3x^4 - 12x^3 + 15x^2 - 6x + 1 \\
 \quad \quad \quad \underline{3x^4 - 12x^3 + 15x^2 - 6x + 1} \\
 \quad \quad \quad \quad 0
 \end{array}$$

Explanation.

 x^2 , root. $3x^4$, divisor. $3x^4 - 6x^3 + 4x^2$, complete divisor. $3(x^2 - 2x)^2 = 3x^4 - 12x^3 + 12x^2$, 2d divisor. }

Annex

 $3x^2 - 6x$

Annex

 $+ 1.$ $3x^4 - 12x^3 + 15x^2 - 6x + 1$, 2d completed divisor.

2. $a^3 + 3a^2 + 3a + 1.$
3. $x^3 + 6x^2 + 12x + 8.$
4. $8m^3 - 12m^2 + 6m - 1.$
5. $64a^3 - 144a^2b + 108ab^2 - 27b^3.$
6. $x^6 + 6x^5 - 40x^3 + 96x^2 - 96x - 64.$
7. $y^6 - 1 + 5y^3 - 3y^5 - 3y.$
8. $15x^4 - 6x - 6x^5 + 15x^2 + 1 + x^6 - 20x^3.$
9. $9x^3 - 21x^2 - 36x^5 + 8x^6 - 9x + 42x^4 - 1.$
10. $8a^6 - 12a^5 - 54a^4 + 59a^3 + 135a^2 - 75a - 125.$

Cube Root of Numbers.

Separating the following numbers into three-figure periods as far as possible, we have :

$$\begin{array}{lll}
 1'. & 1'000. & 1'000'000. \\
 729'. & 980'001. & 998'000'001.
 \end{array}$$

Using the radical sign, we have :

One period.	Two periods.	Three periods.
$\sqrt[3]{1'} = 1.$	$\sqrt[3]{1'000} = 10.$	$\sqrt[3]{1'000'000} = 100.$
$\sqrt[3]{729} = 9.$	$\sqrt[3]{980'001} = 99.$	$\sqrt[3]{998'000'001} = 999.$

Obviously one figure in the cube gives but one figure in the root; two periods in the cube, two figures in the root; three periods in the cube, three figures in the root.

Hence the principle :

The number of figures in the cube root equals the number of three-figure periods into which the number can be pointed off, beginning at units.

NOTE.—The period on the extreme left may contain only one or two figures.

Letting t represent the tens of a root and u the units, we have the formula $(t + u)^3 = t^3 + 3t^2u + 3tu^2 + u^3$.

By means of the above principle and formula, the extraction of the cube root of numbers is readily explained.

Extract the cube root of 46656.

	Explanation.	Root.
Pointing off, we have	$46'656 = t^3 + 3t^2u + 3tu^2 + u^3.$	$\begin{pmatrix} t & u \\ 3 & 6. \end{pmatrix}$
Greatest cube in 46	$= \underline{27} = t^3$	
$\sqrt[3]{t^3} = \sqrt[3]{27} = 3$	$19656 = 3t^2u + 3tu^2 + u^3 =$	
$3t^2u \div 3t^2 = u$	$(3t^2 + 3tu + u^2) u.$	
$\therefore 196 \div 3 \times 3^2 \text{ or } 27$	Complete divisor.	
$= 6 \text{ units.}$		

To complete the divisor

we must annex to $3t^2$

both $3tu$ and u^2 .

$$3t^2 = 27$$

$$3tu = 54$$

$$u^2 = \underline{36}$$

$$3276 \times 6 = \underline{19656}$$

Brief directions are :

1. Point off into periods of 3 figures each What ?
2. Extract root Of what ?
3. Subtract What ?
4. Divide by 3 times the square of the root. Divide what ?
5. Add, and add again What ?
6. Multiply By what ?
7. Subtract What ?
8. Use again (4), (5), (6), and (7) when necessary.

EXERCISES.

1. Find the cube root of:

1. 614125.

Explanation.

Pointing off the number, we have 614'125(85.

Extracting root of greatest cube, 512,...

We have 8.

Subtracting, we have 102 125

Dividing by 3 times 8^2 , we have $102125 \div 19200 = 5$.

Annexing $8 \times 5 \times 3$ and 5^2 ,

we have 19200
1200
25

20425, complete divisor.

Multiplying 20425 by 5, we have 102 125

Subtracting, we have 0

The process, freed from explanation, stands thus :

	614125(85
19200	<u>512</u>
1200	102125
<u>25</u>	<u>102125</u>
20425	0

2. 74,088.

4. 32,768.

3. 15,625.

5. 103,823.

6. 592,704.

9. 145,531,576.

7. 1,860,867.

10. 264,609,288.

8. 34,328,125.

11. 1,879,080,904.

Cube Root of Common and Decimal Fractions.

1. What is the cube root of $\frac{1}{27}$?

$$\sqrt[3]{\frac{1}{27}} = \frac{1}{3}; \text{ for } \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = \left(\frac{1}{3}\right)^3 = \frac{1}{27}.$$

FORMULA.

$$\sqrt[3]{\text{Fraction}} = \frac{\sqrt[3]{\text{Numerator}}}{\sqrt[3]{\text{Denominator}}}.$$

2. What is the cube root of .8?

Not .2, for $.2 \times .2 \times .2 = .008$.

$$\sqrt[3]{.8} = \sqrt[3]{.800000}.$$

By rule: .800'000 (.92 +

	729
24300	71000
540	49688
4	21312, Rem.
24844	

Hence $\sqrt[3]{.8} = .92 +$.

In such cases annex ciphers, and make periods *from the point toward the right*.

3. Find the cube root of:

1. $\frac{8}{27}$.

4. $\frac{64}{512}$.

7. $\frac{19683}{166375}$.

10. $\frac{68921}{148877}$.

2. $\frac{1}{216}$.

5. $\frac{125}{729}$.

8. $\frac{12167}{32768}$.

11. $\frac{274625}{438976}$.

3. $\frac{27}{343}$.

6. $\frac{1728}{46656}$.

9. $\frac{39304}{46656}$.

12. $\frac{551368}{614125}$.

4. Find the cube root of:

1. .008.

3. .8.

5. .25.

2. .08.

4. .125.

6. 2.197.

- | | | |
|-------------|----------------|----------------|
| 7. 9.261. | 10. 6. | 13. 41.063625. |
| 8. 185.193. | 11. 7. | 14. .000001. |
| 9. .1. | 12. 34.965783. | 15. .0000001. |

Suggestion: $\sqrt[3]{.08} = \sqrt[3]{.080} = .4$, etc. Find two more places.

5. Find the cube root of:

- | | | | | |
|---------------------|--------------------|--------------------|--------------------------|----------------------------|
| 1. $\frac{3}{4}$. | 3. $\frac{3}{9}$. | 5. $\frac{2}{3}$. | 7. $\frac{1728}{1898}$. | 9. $\frac{5760}{7000}$. |
| 2. $\frac{9}{13}$. | 4. $\frac{5}{4}$. | 6. $\frac{5}{6}$. | 8. $\frac{1492}{1776}$. | 10. $\frac{8000}{63360}$. |

Suggestion: $\frac{3}{4} = .75$; $\sqrt[3]{.750} = \text{what?}$

Or, $\frac{3}{4} = \frac{6}{8}$; $\sqrt[3]{\frac{6}{8}} = \frac{\sqrt[3]{6}}{2} = \text{what?}$

We have found that an exponent and an index, as signs of operation, are diametrically opposite in effect.

Since $(2^3)^2 = 2^6$,

we have $2 \times 3 \sqrt[2]{2^6} = \sqrt[3]{2^6} = 2$.

In general, $\sqrt[mn]{a^{mn}} = \sqrt[n]{a^n} = a$.

That is,

The mnth root of a quantity is equal to the nth root of the nth root of that quantity.

$\sqrt[4]{a} = \sqrt[2]{\sqrt[2]{a}}$; $\sqrt[6]{a} = \sqrt[3]{\sqrt[2]{a}}$; $\sqrt[8]{a} = \sqrt[2]{\sqrt[4]{a}}$, and so on.

Find the 4th roots of:

- $81x^4 + 108x^3 + 54x^2 + 12x + 1$.
- $16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4$.
- $1 - 4x + 10x^2 - 16x^3 + 19x^4 - 16x^5 + 10x^6 - 4x^7 + x^8$.

Find the 6th root of:

- $15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6 + 6x + 1$.

Find the 8th root of:

- $a^8 - 8a^7b + 28a^6b^2 - 56a^5b^3 + 70a^4b^4 - 56a^3b^5 + 28a^2b^6 - 8ab^7 + b^8$.

EXPONENTS.

Since $\sqrt[3]{a^3} = a^{\frac{3}{3}}$, $\sqrt[3]{a^5} = a^{\frac{5}{3}}$; and, in general, since $\sqrt[m]{a^m} = a^{\frac{m}{m}}$, $\sqrt[m]{a^x} = a^{\frac{x}{m}}$. Hence, an exponent may take the fractional form, the numerator expressing a power, and the denominator a root.

The sixth root of a fifth power may be written $\sqrt[6]{a^5}$ or $a^{\frac{5}{6}}$.

EXERCISES.

Express with fractional exponents :

- | | | |
|----------------------|--------------------|-------------------------------------|
| 1. $\sqrt[3]{x^4}$. | 4. $\sqrt[4]{n}$. | 7. $\sqrt[3]{b^4} \sqrt[5]{x}$. |
| 2. $\sqrt[5]{y^3}$. | 5. $\sqrt[3]{p}$. | 8. $\sqrt[4]{y^5} \sqrt[3]{z^4}$. |
| 3. $\sqrt{m}$. | 6. $\sqrt{c^5}$. | 9. $3a \sqrt[m]{b} \sqrt[n]{c^m}$. |

Express with the radical sign :

- | | | | |
|--|--|---|---|
| 1. $x^{\frac{1}{3}}$. | 2. $a^{\frac{3}{5}}$. | 3. $3b^{\frac{1}{4}}$. | 4. $xy^{\frac{3}{4}}$. |
| 5. $a^{\frac{3}{4}} b^{\frac{2}{3}}$. | 6. $x^{\frac{3}{5}} y^{\frac{5}{3}}$. | 7. $(5m)^{\frac{4}{7}} n^{\frac{7}{2}}$. | 8. $ab^{\frac{1}{3}} c^{\frac{4}{5}} d^{\frac{7}{3}}$. |

Find the value of :

- | | | | | |
|-------------------------|----------------------------|--------------------------|-------------------------|----------------------------|
| 1. $4^{\frac{5}{2}}$. | 2. $9^{\frac{5}{2}}$. | 3. $27^{\frac{2}{3}}$. | 4. $27^{\frac{4}{3}}$. | 5. $36^{\frac{3}{2}}$. |
| 6. $49^{\frac{1}{2}}$. | 7. $-1000^{\frac{2}{3}}$. | 8. $100^{\frac{3}{2}}$. | 9. $1^{\frac{1}{9}}$. | 10. $-125^{\frac{5}{3}}$. |

The Value of a^0 and a^{-1} , etc.

$$a^2 = aa \quad (1)$$

$$a^1 = a \quad (2)$$

$$a^0 = 1 \quad (3)$$

Dividing equation (1) by a , we have equation (2); dividing

equation (2) by a , we have equation (3). That is, *Any quantity whose exponent is 0 is equal to 1.*

$$a^0 = 1 \quad (3)$$

$$a^{-1} = \frac{1}{a} \quad (4)$$

$$a^{-2} = \frac{1}{a^2} \quad (5)$$

etc.

Dividing equation (3) by a , we have equation (4); dividing equation (4) by a , we have equation (5). That is, in general,

$$\frac{a^{-m}}{1} = \frac{1}{a^m}. \quad \text{Hence,}$$

PRINCIPLE.

The transfer of a factor from one term of a fraction to the other requires the change of the sign of its exponent.

$$\text{Thus } \frac{a^2b}{x^{-2}y^{-3}} = a^2bx^2y^3 = \frac{1}{a^{-2}b^{-1}x^{-2}y^{-3}} = \frac{x^2y^3}{a^{-2}b^{-1}}, \text{ etc.}$$

EXERCISES.

Change the negative exponents to positive without changing the value of:

1. a^2y^{-3} .

6. $4a^0b^{\frac{1}{2}}c^{-\frac{1}{3}}$.

2. x^0y^{-5} .

7. $3x^{-3}y^{-\frac{4}{5}}$.

3. $p^3q^{-\frac{1}{3}}$.

8. $6x^{-5}b^{-7}c^0$.

4. $5x^2y^{-\frac{1}{2}}$.

9. $7m^{-7}n^7$.

5. $a^{-2}b^{-3}$.

10. $x^{-\frac{1}{2}}y^{-\frac{2}{3}}z^{-\frac{5}{4}}$.

Any quantity whose exponent is 0 equals what?

Transfer the literal factors of the denominators into the numerators in :

1. $\frac{1}{m}$

6. $\frac{xy^3}{zu^{\frac{1}{3}}}$

2. $\frac{x^2}{a^3}$

7. $\frac{6x^3}{3y^2z^4}$

3. $\frac{5}{a^{-2}}$

8. $\frac{x^3}{3a^{\frac{3}{4}}b^{\frac{4}{7}}}$

4. $\frac{a^0}{3x^{\frac{4}{5}}}$

9. $\frac{5x}{5m^{-5}n^{-\frac{5}{5}}}$

5. $\frac{4c}{a^3b^{-2}}$

10. $\frac{m^0n^0}{m^{\frac{4}{5}}n^{-\frac{5}{7}}x^0}$

Transfer the literal factors of the numerators into the denominators in :

1. $\frac{m^3}{3}$

6. $\frac{6x^{-6}y}{c^{\frac{1}{3}}}$

2. $\frac{4x^{\frac{1}{4}}}{4a}$

7. $\frac{a^{-\frac{3}{5}}b^{\frac{1}{5}}}{m}$

3. $\frac{5x^{-4}}{5}$

8. $a^{-1}b^2c^3d^0$

4. $\frac{3d^{-\frac{4}{5}}}{7}$

9. $\frac{5a^{-3}b^{-\frac{3}{2}}}{7c^5}$

5. $\frac{4b^{\frac{6}{7}}}{1}$

10. $x^0y^{-1}z^2u^{-3}$

Since $a^2 \times a^3 = a^{2+3} = a^5$, we have, for like reason, $a^3 \times a^{-2} = a$, and $a^2 \times a^{-3} = a^{-1} = \frac{1}{a}$.

In general, $a^m \times a^{-n} = \frac{a^m}{a^n} = a^{m-n}$.

Find the value of:

1. $a^3 \times a^{-1}$

6. $5x^{-\frac{3}{5}} \times 6 \sqrt[5]{x^3}$

2. $a \times a^{-3}$

7. $7a^{-5}b^{\frac{2}{3}}ab^{-1}$

3. $a^{-3}a^3$

8. $9x^2y^{-\frac{5}{3}} \times \frac{x^{-2}y^{\frac{7}{3}}}{3}$

4. $b^{\frac{3}{4}} \times b^{-\frac{1}{4}}$

9. $11\sqrt[5]{a} \times 4\sqrt[4]{a^2}$

5. $3c^{\frac{1}{2}} \times c^{-\frac{3}{2}}$

10. $\frac{13xy}{a^{\frac{1}{3}}b^{-3}} \times \frac{2}{a^{-2}b^{\frac{1}{3}}}$

Multiply $m + 3m^{\frac{1}{3}} - 4m^{\frac{2}{3}}$ by $3 - 5m^{-\frac{2}{3}} - 7m^{-\frac{1}{3}}$.

Process.

$$\begin{array}{r}
 m + 3m^{\frac{1}{3}} - 4m^{\frac{2}{3}} \\
 3 - 5m^{-\frac{2}{3}} - 7m^{-\frac{1}{3}} \\
 \hline
 3m + 9m^{\frac{1}{3}} - 12m^{\frac{2}{3}} \\
 - 5m^{\frac{1}{3}} - 15m^{-\frac{1}{3}} + 20 \\
 - 7m^{\frac{2}{3}} - 21 + 28m^{\frac{1}{3}} \\
 \hline
 3m + 4m^{\frac{1}{3}} - 19m^{\frac{2}{3}} - 15m^{-\frac{1}{3}} + 28m^{\frac{1}{3}} - 1 \\
 = 3m + 4\sqrt[3]{m} - 19\sqrt[3]{m^2} - 15\sqrt[3]{\frac{1}{m}} + 28\sqrt[3]{m} - 1.
 \end{array}$$

Multiply:

1. $m^{\frac{1}{3}} + n^{\frac{1}{3}}$ by $m^{\frac{1}{3}} - n^{\frac{1}{3}}$

2. $m^{\frac{1}{3}} + n^{\frac{1}{3}}$ by $m^{\frac{1}{3}} + n^{\frac{1}{3}}$

3. $m^{\frac{1}{3}} - n^{\frac{1}{3}}$ by $m^{\frac{1}{3}} - n^{\frac{1}{3}}$

4. $a^2 - 2 + a^{-2}$ by $a^2 + 2 + a^{-2}$

5. $x^{-2} + x^{-1}y^{-1} + y^{-2}$ by $x^{-2} - x^{-1}y^{-1} + y^{-2}$

6. $x^{\frac{2}{3}} - x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}}$ by $x^{\frac{1}{3}} + y^{\frac{1}{3}}$

7. $x^{-4} - x^{-2}y^{-4} + y^{-8}$ by $x^{-2} + y^{-4}$

8. $a^{\frac{1}{2}} + a^{\frac{1}{4}}b^{\frac{1}{4}} + b^{\frac{1}{2}}$ by $a^{\frac{1}{2}} - a^{\frac{1}{4}}b^{\frac{1}{4}} + b^{\frac{1}{2}}$

$$9. x^{\frac{2}{3}}y^{-\frac{1}{2}} + 2x^{\frac{1}{3}} - 3y^{\frac{1}{2}} \text{ by } 2y^{-\frac{1}{2}} - 4x^{-\frac{1}{3}} - 6x^{-\frac{2}{3}}y^{\frac{1}{2}}.$$

$$10. 4a^{\frac{3}{4}}b^{-1} + a^{\frac{1}{4}} - 2a^{-\frac{1}{4}}b \text{ by } 7a^{\frac{1}{4}}b^{-1} - 2a^{-\frac{1}{4}} - 3a^{-\frac{3}{4}}b.$$

Since $\frac{a^5}{a^3} = a^{5-3} = a^2$, and $\frac{a^3}{a^5} = a^{3-5} = a^{-2}$, for like reason we

have $\frac{a^1}{a^{\frac{2}{3}}} = a^{1-\frac{2}{3}} = a^{+\frac{1}{3}}$, and $\frac{a^{\frac{2}{3}}}{a} = a^{\frac{2}{3}-1} = a^{-\frac{1}{3}}$

In general, $\frac{a^m}{a_n^m} = a^{m-\frac{m}{n}} = a^{\frac{mn-m}{n}}$, and $\frac{a_n^m}{a^m} = a^{\frac{m}{n}-m} = a^{\frac{m-mn}{n}}$.

EXERCISES.

Divide :

$$1. x^3 \text{ by } a^{-2}.$$

$$6. m^3 \text{ by } \sqrt[3]{m^{-2}}.$$

$$2. x \text{ by } x^4.$$

$$7. a^{\frac{1}{3}} \text{ by } \sqrt[5]{a^4}.$$

$$3. x^{\frac{4}{9}} \text{ by } x^{\frac{4}{5}}.$$

$$8. 16x \text{ by } 4x^{-1} \sqrt[4]{y}.$$

$$4. x^{-\frac{1}{3}} \text{ by } x^{-\frac{3}{5}}.$$

$$9. 7a^{-1}x^{\frac{2}{3}} \text{ by } 3\frac{1}{2}ax^{-\frac{1}{5}}.$$

$$5. 5d^{-2} \text{ by } \sqrt[3]{k^5}.$$

$$10. x^{-n} \text{ by } x^{\frac{1}{n}}.$$

$$11. m - n \text{ by } m^{\frac{1}{3}} - n^{\frac{1}{3}}.$$

Process.

$$m^{\frac{1}{3}} - n^{\frac{1}{3}} \big) m - n (m^{\frac{2}{3}} + m^{\frac{1}{3}}n^{\frac{1}{3}} + n^{\frac{2}{3}}.$$

$$\underline{m - m^{\frac{2}{3}}n^{\frac{1}{3}}}$$

$$+ m^{\frac{2}{3}}n^{\frac{1}{3}} - n$$

$$\underline{+ m^{\frac{2}{3}}n^{\frac{1}{3}} - m^{\frac{1}{3}}n^{\frac{2}{3}}}$$

$$m^{\frac{1}{3}}n^{\frac{2}{3}} - n$$

$$\underline{m^{\frac{1}{3}}n^{\frac{2}{3}} - n}$$

$$0$$

12. $a + b$ by $a^{\frac{1}{3}} + b^{\frac{1}{3}}$.

13. $a - b$ by $a^{\frac{1}{4}} - b^{\frac{1}{4}}$.

14. $x - 3x^{\frac{2}{3}} + 3x^{\frac{1}{3}} - 1$ by $x^{\frac{1}{3}} - 1$.

15. $21x + x^{\frac{2}{3}} + x^{\frac{1}{3}} + 1$ by $3x^{\frac{1}{3}} + 1$.

16. $x^{\frac{2}{3}} - 4x^{\frac{1}{3}} + 1 + 6x^{-\frac{1}{3}}$ by $x^{\frac{1}{3}} - 2$.

17. $9x - 12x^{\frac{1}{2}} - 2 + 4x^{-\frac{1}{2}} + x^{-1}$ by $3x^{\frac{1}{2}} - 2 - x^{-\frac{1}{2}}$.

18. $a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}$ by $a^{\frac{1}{3}} - a^{\frac{1}{6}}b^{\frac{1}{6}} + b^{\frac{1}{3}}$.

19. $21a^{3x} + 20 - 27a^x - 26a^{2x}$ by $3a^x - 5$.

20. $5b^{\frac{2}{3}} - 6b^{\frac{1}{3}} - 4b^{-\frac{2}{3}} - 4b^{-\frac{1}{3}} - 5$ by $b^{\frac{1}{3}} - 2b^{-\frac{1}{6}}$.

Square :

1. $a^{\frac{2}{3}} + b^{\frac{1}{2}}$.

2. $a^{-\frac{1}{3}} - b^{-\frac{3}{5}}$.

3. $2x^{\frac{1}{2}}y^{\frac{1}{2}} + x^{-\frac{1}{2}}y^{-\frac{1}{2}}$.

4. $x^0 - y^m$.

5. $\frac{1}{m^{-1}} - \frac{m^{-1}}{1}$.

Cube :

1. $a^{-\frac{2}{3}} - b^{-\frac{3}{2}}$.

2. $2x - 3x^2$.

3. $x^{\frac{1}{2}} - 2x^{\frac{3}{4}}$.

4. $\frac{1}{2}x^{-\frac{1}{2}} - 2x^{\frac{1}{4}}$.

5. $\sqrt[3]{x} - \sqrt[3]{y}$.

Write the value of :

1. $(x^{\frac{a}{2}} - \frac{1}{2}x^{-a})^2$.

2. $(x^{3a} + 8) \div (x^a + 2)$.

3. $(x^{5n} + 32) \div (x^n + 2)$.

4. $(2x^{\frac{1}{3}} + 4 + 3x^{-\frac{1}{3}})(2x^{\frac{1}{3}} + 4 - 3x^{-\frac{1}{3}})$.

5. $(3x^{\frac{1}{2}} - 2 - x^{-\frac{1}{2}})^2$.

MISCELLANEOUS EXERCISES.

Whether m and n are integral or fractional, positive or negative, the following are true :

$$1. a^m \times a^n = a^{m+n}.$$

$$2. \frac{a^m}{a^n} = a^{m-n}.$$

$$3. (ab)^m = a^m b^m.$$

$$4. (a^m)^n = a^{mn}.$$

$$5. \sqrt[n]{a^m b^n} = ab^{\frac{n}{m}}.$$

It is also to be remembered that $a^0 = 1$ and $a^{-m} = \frac{1}{a^m}$.

Transfer literal factors into the numerators :

$$\frac{2a^{\frac{1}{3}}m^{\frac{1}{2}}}{5c\sqrt{m}}; \frac{3x^{\frac{1}{4}}y\sqrt{(cd^3)}}{x^2y\sqrt[3]{d}}; \frac{1}{\sqrt[3]{(xy^2cd^{\frac{1}{4}})}}; \frac{m^{\frac{1}{2}}n^{\frac{1}{3}}p^{\frac{1}{4}}}{a^{\frac{1}{2}}n^{\frac{1}{3}}p^{\frac{1}{4}}}.$$

Transfer literal factors into the denominators :

$$\frac{4\sqrt[3]{xy^2z^4}}{7\sqrt{ab^3}}; \frac{5xa}{\sqrt{d}}; \frac{mn^2c^{\frac{1}{3}}}{3b^3x\sqrt{y}}; \frac{6pq}{7mn}.$$

Simplify by performing the operations indicated :

$$1. \frac{a^{2n} - b^{2n}}{a^n + b^n} = ?$$

$$5. (a^2b^{-\frac{1}{2}})^{-\frac{1}{4}}.$$

$$2. \sqrt[3]{\frac{1}{x^3y^2}} (x^2y^{\frac{2}{3}})$$

$$6. \frac{(a+x)^{\frac{1}{3}}\sqrt{b}}{\sqrt{(a+x)^{\frac{1}{2}}}}.$$

$$3. (a^m)^{m-\frac{1}{m}}.$$

$$4. \frac{x^{a+b} x^{a-b} x^{c-2a}}{x^{c-a}}.$$

$$7. \left(\frac{1}{x^3y^2}\right)^{\frac{1}{6}} \cdot (x^2)^{\frac{1}{5}} \cdot y^{\frac{1}{3}}.$$

$$8. \{- (a^3)^{\frac{1}{2}}\}^{-\frac{1}{3}} \cdot \{- (-a)^{-3}\}^{\frac{1}{2}}.$$

9. $(-(x^3)^{\frac{1}{2}})^{-\frac{1}{3}}$.
10. $[\{(a^{-m})^{-n}\}^{-p}] \div a^{2m}$.
11. $(a^2b^{-\frac{1}{2}}c^{\frac{2}{5}})^{-\frac{1}{4}}$.
12. $[x^{a^2-ab}x^{b^2-ab}]^{\frac{1}{a-b}}$.
13. $(x^a)^{-b} \div (x^{-a})^{-b}$.
14. $[(a+b)^{\frac{1}{3}} - (a-b)^{\frac{1}{3}}]^2$.

Extract the square root of:

$$\begin{array}{r}
 15. \quad \begin{array}{r} 9xy^{-2} + 12y^{-1} + 4x^{-1} \end{array} \bigg| \frac{3x^{\frac{1}{2}}y^{-1} + 2x^{-\frac{1}{2}}}{9xy^{-2}} \\
 \hline
 \begin{array}{r} 6x^{\frac{1}{2}}y^{-1} + 2x^{-\frac{1}{2}} \end{array} \bigg| \begin{array}{r} 12y^{-1} + 4x^{-1} \\ 12y^{-1} + 4x^{-1} \end{array} \\
 \hline
 0
 \end{array}$$

16. $x^{-1} - 4x^{-\frac{1}{2}}y^{\frac{1}{2}} + 10y - 12x^{\frac{1}{2}}y^{\frac{3}{2}} + 9xy^2$.
17. $9a^{-3} - 30a^{-\frac{5}{2}}b + 13a^{-2}b^2 + 20a^{-\frac{3}{2}}b^3 + 4a^{-1}b^4$.

Extract the cube root of:

18. $a^{-\frac{3}{2}} - 3a^{-1}b^{\frac{2}{3}} + 3a^{-\frac{1}{2}}b^{\frac{4}{3}} - b^2$.
19. $x - 3x^{\frac{2}{3}} + 9x^{\frac{1}{3}} - 13 + 18x^{-\frac{1}{3}} - 12x^{-\frac{2}{3}} + 8x^{-1}$.

Resolve into binomial factors:

20. $a - b$.
21. $c^{-1} - d^{-1}$.
22. $m^{\frac{1}{3}} - n^{\frac{1}{3}}$.
23. $p^m - q^n$.
24. $m^{-a} - n^{-b}$.

Resolve into trinomial factors:

25. $m + m^{\frac{1}{2}}n^{\frac{1}{2}} + n$.

$$26. a^{-4} + a^{-2}b^{-2} + b^{-4}.$$

$$27. a^2 + ab + b^2.$$

$$28. x^3 - 8x^{-1}.$$

$$29. a^{\frac{6}{5}} + a^{\frac{3}{5}}b^{\frac{3}{5}} + b^{\frac{1}{5}}.$$

RADICALS.

A Radical is a quantity directly affected by the radical sign ; as $\sqrt{m}$.

Quantities thus affected are *rational* or *irrational* : rational when the root can be exactly obtained, as $\sqrt[3]{27}$ and $\sqrt{a^2 + 2ab + b^2}$; irrational or *surd* when the root indicated *cannot* be exactly obtained, as $\sqrt[3]{2}$ and $\sqrt{a + b}$.

The *degree* of a radical is denoted by the index of its radical sign, or by the denominator of its fractional exponent ; as $\sqrt[5]{ab}$, or $(ab)^{\frac{1}{5}}$, a radical of the fifth degree.

Similar radicals have the same degree and the same quantities under the radical sign ; as $\sqrt[5]{a + b}$ and $5\sqrt[5]{a + b}$. The factor 5, before the sign, is called the coefficient of the radical.

Since $\sqrt[m]{a^m b^n} = ab_m^{\frac{n}{m}}$, it is plain that a radical may be simplified when it contains a factor of which the given root may be taken.

1. Simplify, or reduce, $\sqrt{8a^2bc}$.

Process.

$$\sqrt{8a^2bc} = \sqrt{4a^2 \cdot 2bc} = 2a\sqrt{2bc}.$$

2. Reduce $3\sqrt[3]{189}$.

Process.

$$3\sqrt[3]{189} = 3\sqrt[3]{27 \cdot 7} = 3 \times 3\sqrt[3]{7} = 9\sqrt[3]{7}.$$

3. Simplify $\sqrt[n]{x^{3n}y^{2n+5}}$.

Process.

$$\sqrt[n]{x^{3n}y^{2n+5}} = \sqrt[n]{x^{3n} \cdot y^{2n+5}} = x^3 \sqrt[n]{y^{2n+5}}.$$

EXERCISES.

1. $\sqrt{288}$.

9. $\sqrt[mn]{x^{2mn}y^{3n}}$.

2. $\sqrt{147}$.

10. $\sqrt[6]{1728} (= 12^3)$.

3. $\sqrt[3]{256}$.

11. $\sqrt[3]{250x^2y^3z^7}$.

4. $\sqrt[3]{432}$.

12. $\sqrt{(a^2 - b^2)(a + b)}$.

5. $3\sqrt{150}$.

13. $\sqrt{20a^2 + 60a + 45}$.

6. $\sqrt{27a^3b^5}$.

14. $\sqrt[p]{x^a + py^{2p}}$.

7. $\sqrt[3]{-108x^4y^3}$.

15. $\sqrt[3]{\frac{3}{4}}$.

8. $\sqrt{a^3 + 2a^2b + ab^2}$.

$$\sqrt[3]{\frac{3}{4}} = \sqrt[3]{\frac{6}{8}} = \sqrt[3]{\frac{1}{8}} \times 6 = \frac{1}{2} \sqrt[3]{6}.$$

16. $\sqrt{\frac{1}{2}}$.

19. $\sqrt[3]{\frac{77}{8}}$.

22. $\sqrt[4]{\frac{b^4}{16a^3}}$.

17. $\sqrt{\frac{1}{3}}$.

20. $\sqrt{\frac{20c^2}{9a^2b}}$.

23. $\sqrt[2n+1]{\frac{x^{2n+1}}{y^3}}$.

18. $\sqrt[3]{\frac{5}{9}}$.

21. $\sqrt[3]{\frac{27x^4}{a^2}}$.

24. $\frac{2}{5} \sqrt{12\frac{1}{2}}$.

Since $\sqrt[m]{a^m b^n} = ab^{\frac{n}{m}}$, the factors a and $b^{\frac{n}{m}}$ can be returned beneath the sign $\sqrt[m]{}$ by involving them to the m th power.

1. Introduce the coefficient of $2\sqrt[3]{5}$ under the radical sign. $2\sqrt[3]{5} = \sqrt[3]{2^3 \cdot 5} = \sqrt[3]{8 \cdot 5} = \sqrt[3]{40}$.

2. Introduce the coefficient of $5a\sqrt[3]{2x^2}$ under the radical sign.

$$5a\sqrt[3]{2x^2} = \sqrt[3]{5^3 a^3 \cdot 2x^2} = \sqrt[3]{125 a^3 \cdot 2x^2} = \sqrt[3]{250 a^3 x^2}.$$

$\sqrt[3]{40}$ and $\sqrt[3]{250 a^3 x^2}$ may be called *entire radicals*.

EXERCISES.

Render the following radicals or surds entire :

1. $7\sqrt{2}$.

8. $\frac{x}{a^2} \sqrt{\frac{3a^3}{x}}$.

2. $13\sqrt{5}$.

9. $\frac{2x}{3a} \sqrt[3]{\frac{27a^4}{x^2}}$.

3. $3\sqrt[3]{3}$.

4. $9\sqrt[3]{5}$.

10. $\frac{2x}{y} \sqrt[4]{\frac{y^4}{8x^3}}$.

5. $\frac{2}{7} \sqrt{\frac{7}{9}}$.

6. $\frac{2xy}{c} \sqrt{\frac{10c^2}{9x^2y}}$.

11. $x \sqrt{\frac{y^2}{x^{n-2}}}$.

7. $\frac{5a}{b} \sqrt{\frac{x^2b^3}{a^2}}$.

12. $(x+y) \sqrt{\frac{x-y}{x+y}}$.

Addition and Subtraction of Radicals.

The addition or subtraction of radicals requires that, when they are not similar, they be made similar.

1. Find the value of $3\sqrt{45} + \sqrt{20}$.

$$3\sqrt{45} = 3\sqrt{9 \times 5} = 3 \times 3\sqrt{5} = 9\sqrt{5}$$

$$\sqrt{20} = \sqrt{4 \times 5} = 2\sqrt{5}$$

$11\sqrt{5}$, sum required.

2. Find the value of $\sqrt{98a} - \sqrt{50a}$.

$$\sqrt{98a} = \sqrt{49 \times 2a} = 7\sqrt{2a}$$

$$\sqrt{50a} = \sqrt{25 \times 2a} = 5\sqrt{2a}$$

$2\sqrt{2a}$, difference required.

EXERCISES.

Find the value of :

1. $3\sqrt{45} + 7\sqrt{5}$.

2. $7\sqrt{5} - \sqrt{20}$.

3. $4\sqrt{63} + 5\sqrt{7}$.

4. $4\sqrt{63} - 8\sqrt{28}$.

5. $\sqrt{44} + 2\sqrt{99}$.

6. $\sqrt{44} - 5\sqrt{176} + 2\sqrt{99}$.

7. $2\sqrt{363} - 5\sqrt{243} + \sqrt{192}$.

8. $3\sqrt[3]{875} - 2\sqrt[3]{189}$.

9. $3\sqrt[3]{875} - 7\sqrt[3]{56}$.

10. $5\sqrt{24} - 2\sqrt{54} + \sqrt{6}$.

11. $\sqrt{\frac{3}{5}} - \sqrt{\frac{5}{27}}$.

12. $4\sqrt{\frac{7}{3}} + 12\sqrt{\frac{3}{4}}$.

Multiplication of Radicals.

Since $(ab)^m = a^m b^m$, we may write

$$\sqrt{3} \times \sqrt{7} = \sqrt{21}.$$

1. Multiply $3\sqrt{6}$ by $2\sqrt{3}$.

$$3\sqrt{6} \times 2\sqrt{3} = 6\sqrt{18} = 6\sqrt{9 \times 2} = 6 \times 3\sqrt{2} = 18\sqrt{2},$$

Ans.

2. Multiply $\sqrt[3]{a+b}$ by $\sqrt[3]{a-b}$.

$$\sqrt[3]{a+b} \times \sqrt[3]{a-b} = \sqrt[3]{a^2-b^2}. \quad \text{Ans.}$$

3. Multiply $a\sqrt{b^3} \times b^2\sqrt{a}$.

$$a\sqrt{b^3} \times b^2\sqrt{a} = ab^2\sqrt{ab^3} = ab^3\sqrt{ab}. \quad \text{Ans.}$$

EXERCISES.

Find the value of:

1. $2\sqrt{15} \times 3\sqrt{5}$.

6. $\frac{3}{x}\sqrt{\frac{a^2}{x}} \times \frac{4}{3}\sqrt{\frac{x^3}{2a^4}}$.

2. $8\sqrt{12} \times 3\sqrt{24}$.

7. $5\sqrt[5]{3} \times 5\sqrt{2}$.

3. $\sqrt{x+2} \times \sqrt{x-2}$.

8. $\sqrt{3} \times \sqrt[3]{2} \times \sqrt[3]{\frac{1}{6}}$.

4. $\sqrt[3]{168} \times \sqrt[3]{147}$.

9. $\sqrt[3]{\frac{1}{2}} \times \sqrt{\frac{2}{3}} \times \sqrt{\frac{3}{4}}$.

5. $5\sqrt[3]{128} \times 2\sqrt[3]{432}$.

10. $\frac{2}{3}\sqrt{\frac{1}{8}} \times \frac{3}{4}\sqrt{\frac{7}{10}}$.

$\sqrt{mn} \times \sqrt{m'n} = \sqrt{m^2n^2} = mn$: hence, to multiply a radical quantity of the second degree by itself, it is simply necessary to remove the radical sign: $(\sqrt{a+b})(\sqrt{a+b}) = a+b$.

1. Multiply $3\sqrt{x-5}$ by $2\sqrt{x}$.

Process.

$$\begin{array}{r} 3\sqrt{x-5} \\ 2\sqrt{x} \\ \hline 6x - 10\sqrt{x}. \end{array}$$

2. Multiply $2\sqrt{5} + 3\sqrt{x}$ by $\sqrt{5} - \sqrt{x}$.

Process.

$$\begin{array}{r} 2\sqrt{5} + 3\sqrt{x} \\ \sqrt{5} - \sqrt{x} \\ \hline 10 + 3\sqrt{5x} \\ - 2\sqrt{5x} - 3x \\ \hline 10 + \sqrt{5x} - 3x. \end{array}$$

EXERCISES.

Find the value of:

1. $(\sqrt{x+y} - 1) \times \sqrt{x+y}$.
2. $(\sqrt{7} + 5\sqrt{3})(2\sqrt{7} - 4\sqrt{3})$.
3. $(\sqrt{x+1} - 2\sqrt{x})(2\sqrt{x+1} + \sqrt{x})$.
4. $(\sqrt{x+1} + \sqrt{x-1})(\sqrt{x+1} - \sqrt{x-1})$.
5. $(a - \sqrt{a+x} + \sqrt{a})(\sqrt{a+x} - \sqrt{a+x})$.

Division of Radicals.

1. Divide
- $6\sqrt{14}$
- by
- $2\sqrt{21}$
- .

$$\frac{6\sqrt{14}}{2\sqrt{21}} = 3\sqrt{\frac{14}{21}} = 3\sqrt{\frac{2}{3}} = 3 \times \frac{1}{3}\sqrt{6} = \sqrt{6}. \quad \text{Ans.}$$

2. Divide
- $\frac{3\sqrt{11}}{2\sqrt{98}}$
- by
- $\frac{5}{7\sqrt{22}}$
- .

$$\frac{3\sqrt{11}}{2\sqrt{98}} \times \frac{7\sqrt{22}}{5} = \frac{21}{10} \sqrt{\frac{242}{98}} = \frac{21}{10} \sqrt{\frac{121}{49}} = \frac{21}{10} \times \frac{11}{7} = \frac{33}{10} = 3\frac{3}{10}. \quad \text{Ans.}$$

EXERCISES.

Find the value of:

- | | |
|--------------------------------------|---|
| 1. $\sqrt{54} \div \sqrt{3}$. | 6. $\sqrt[5]{4} \div \sqrt[4]{6}$. |
| 2. $\sqrt{25d^3} \div \sqrt{d}$. | 7. $\sqrt[5]{24} \div \sqrt[3]{4}$. |
| 3. $\sqrt{18a^6} \div \sqrt{6a^2}$. | 8. $\sqrt[3]{8a} \div \sqrt[4]{4a}$. |
| 4. $\sqrt{12} \div \sqrt[3]{6}$. | 9. $\sqrt[3]{6a^2b} \div \sqrt[5]{12a^3b^2}$. |
| 5. $\sqrt[3]{36} \div \sqrt{12}$. | 10. $\sqrt[6]{24a^3b^2c^5} \div \sqrt[4]{4a^2bc^3}$. |

Involution and Evolution of Radicals.

1. Raise
- $\sqrt[3]{m}$
- to the 6th power.

$$(\sqrt[3]{m})^6 = (m^{\frac{1}{3}})^6 = m^{\frac{6}{3}} = m^2.$$

2. Raise $\left(\sqrt{\frac{a}{3}}\right)$ to the 3d power.

$$\left(\sqrt{\frac{a}{3}}\right)^3 = \left(\left(\frac{a}{3}\right)^{\frac{1}{2}}\right)^3 = \left(\frac{a}{3}\right)^{\frac{3}{2}} = \sqrt{\frac{a^3}{27}} = \frac{a}{3} \sqrt{\frac{a}{3}} = \frac{a}{9} \sqrt{3a}.$$

EXERCISES.

Find the value of:

- | | |
|----------------------------|--------------------------------------|
| 1. $(\sqrt[3]{6})^5$. | 6. $(\sqrt{2} + \sqrt{3})^2$. |
| 2. $(\sqrt[5]{12})^3$. | 7. $(\sqrt{m} + \sqrt{n})^2$. |
| 3. $(\sqrt[3]{x^2a})^6$. | 8. $(x + \sqrt{y})^3$. |
| 4. $(\sqrt[6]{36})^3$. | 9. $(\sqrt[3]{1} - \sqrt[3]{2})^3$. |
| 5. $(\sqrt[4]{x} + y)^2$. | 10. $(\sqrt{10} + 7)^2$. |

Extract the cube root of $8\sqrt[4]{27}$.

$$(8\sqrt[4]{27})^{\frac{1}{3}} = 8^{\frac{1}{3}} \times 27^{\frac{1}{3} \times \frac{1}{4}} = 2 \times 3^{\frac{1}{4}} = 2\sqrt[4]{3}.$$

Find the 5th root of $32a^{10}\sqrt[4]{a^{10}b^5}$.

$$(32a^{10})^{\frac{1}{5}} (a^{10}b^5)^{\frac{1}{4} \times \frac{1}{5}} = 2a^2 (a^2b)^{\frac{1}{4}} = 2a^2 \sqrt[4]{a^2b}.$$

EXERCISES.

Find the value of:

- | | |
|--|--|
| 11. $(\frac{1}{16} \sqrt[3]{256})^{\frac{1}{4}}$. | 16. $\sqrt{\frac{2}{\sqrt[3]{2}}}$. |
| 12. $\sqrt[3]{125^{\frac{1}{2}}}$. | 17. $\sqrt[3]{27a^3} \sqrt{\frac{x^3}{y^6}}$. |
| 13. $\sqrt[5]{32^{\frac{1}{2}}}$. | 18. $\sqrt[4]{81 \sqrt[3]{x^{\frac{2}{3}}}}$. |
| 14. $\sqrt[3]{\sqrt[5]{27a^3}}$. | 19. $\sqrt{8a^3x} \sqrt[3]{(a-x)^2}$. |
| 15. $\sqrt[n]{\sqrt[m]{a^m}}$. | 20. $\sqrt[3]{8x^4} \sqrt{(x^2 - y^2)^3}$. |

Rationalization.

To rationalize a surd is to free it from its irrational elements. $\sqrt[3]{2}$ may be rationalized by multiplying it by $\sqrt[3]{4}$; for $\sqrt[3]{2} \times \sqrt[3]{4} = \sqrt[3]{8} = 2$; and $\sqrt[n]{m} \times \sqrt[n]{m^{n-1}} = \sqrt[n]{m^n} = m$.

The denominator of a fraction may be rationalized by multiplying both terms of the fraction by the appropriate rationalizing factor.

$$\frac{1}{\sqrt{2} \times \sqrt{2}} = \frac{\sqrt{2}}{2}, \text{ and } \frac{1}{\sqrt{x^5}} = \frac{\sqrt{x}}{\sqrt{x^6}} = \frac{\sqrt{x}}{x^3}. \quad \text{Ans.}$$

Rationalize the denominator of :

1. $\frac{1}{\sqrt{3}}$

6. $\frac{6}{\sqrt[4]{10a^3}}$

2. $\frac{3}{\sqrt{b}}$

7. $\frac{3}{\sqrt[5]{9c^4}}$

3. $\frac{5}{4\sqrt[3]{4}}$

8. $\frac{5ab}{\sqrt[6]{32b^5}}$

4. $\frac{x}{\sqrt[3]{c^2}}$

9. $\frac{\sqrt{1-a}}{\sqrt[3]{(1-a)^2}}$

5. $\frac{n}{b\sqrt{m}}$

10. $\frac{a+b}{\sqrt{a^2-b^2}}$

The denominator may be a binomial composed of radicals of the second degree.

$$\begin{aligned} \frac{8}{\sqrt{7} + \sqrt{3}} &= \frac{8}{\sqrt{7} + \sqrt{3}} \times \frac{\sqrt{7} - \sqrt{3}}{\sqrt{7} - \sqrt{3}} = \frac{8(\sqrt{7} - \sqrt{3})}{7 - 3} \\ &= \frac{8}{4}(\sqrt{7} - \sqrt{3}) = 2(\sqrt{7} - \sqrt{3}). \end{aligned}$$

The rationalizing factor for $\sqrt{7} + \sqrt{3}$ differs therefrom only in change of the connecting sign.

EXERCISES.

Rationalize the denominator of :

$$1. \frac{3}{3 + \sqrt{2}}.$$

$$6. \frac{\sqrt{7} - 1}{\sqrt{7} + 1}.$$

$$2. \frac{4}{\sqrt{3} - 1}.$$

$$7. \frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} - \sqrt{y}}.$$

$$3. \frac{3 + \sqrt{3}}{3 - \sqrt{3}}.$$

$$8. \frac{\sqrt{m-n} + \sqrt{n}}{\sqrt{m-n} - \sqrt{n}}.$$

$$4. \frac{\sqrt{5} - \sqrt{3}}{\sqrt{5} + \sqrt{3}}.$$

$$9. \frac{\sqrt{2x-1} + \sqrt{x+1}}{\sqrt{2x-1} - \sqrt{x+1}}.$$

$$5. \frac{2\sqrt{2} + 2}{2\sqrt{2} - 2}.$$

$$10. \frac{\sqrt{a^2-1} - \sqrt{a^2+1}}{\sqrt{a^2-1} + \sqrt{a^2+1}}.$$

Approximate Values.

Find the approximate value of $\frac{2}{2 - \sqrt{2}}$.

$$\frac{2}{2 - \sqrt{2}} = \frac{2(2 + \sqrt{2})}{4 - 2} = \frac{2(2 + \sqrt{2})}{2} = 2 + \sqrt{2} = 2 + 1.414 \\ = 3.414.$$

The work is diminished by rationalizing the denominator.

EXERCISES.

$$1. \frac{3}{\sqrt{3} + \sqrt{2}}.$$

$$3. \frac{4}{\sqrt{3} - 1}.$$

$$2. \frac{1}{2 - \sqrt{2}}.$$

$$4. \frac{7}{\sqrt[3]{7}}.$$

$$5. \frac{3 - \sqrt{2}}{3 + \sqrt{2}}.$$

$$7. \frac{3 + 2\sqrt{7}}{4\sqrt{7} - 5}.$$

$$6. \frac{\sqrt{3} - \sqrt{2}}{\sqrt{3} + \sqrt{2}}.$$

$$8. \frac{\sqrt{6}}{\sqrt{2.5} + \sqrt{6}}.$$

Imaginary Quantities.

Radical quantities are classified, not only as rational and irrational, but as real and imaginary.

$\sqrt[3]{8}$ is a *rational* quantity because its root can be exactly obtained ; $\sqrt{2}$ is an *irrational* quantity, because its root cannot be exactly obtained.

But $\sqrt[3]{8}$ and $\sqrt{2}$ are both real quantities, because they both have real roots : $\sqrt[3]{8} = 2$; $\sqrt{2} = 1.414 \dots$

$\sqrt{-4} =$ neither $+2$ nor -2 , for both $(+2)^2$ and $(-2)^2 = +4$. Hence, for the reason that $\sqrt{-4} =$ no root, exact or approximate, it is called an *imaginary quantity*.

$\sqrt[3]{-8}$ is a *real quantity*, for its root is -2 .

An Imaginary Quantity is, therefore, an indicated *even* root of a negative quantity.

An imaginary quantity, as $\sqrt{-a^2}$, may be expressed in two factors, thus : $\sqrt{a^2(-1)}$. But $\sqrt{a^2(-1)} = a\sqrt{-1}$. Hence, the general formula for the root of an imaginary quantity is $a\sqrt[n]{-1}$, in which n is any *even positive whole* number, and a is the coefficient of the imaginary factor.

Addition and Subtraction of Imaginaries.

1. Find the sum of $\sqrt{-9}$ and $\sqrt{-25}$.

$$\left. \begin{aligned} \sqrt{-9} &= \sqrt{9(-1)} = 3\sqrt{-1} \\ \sqrt{-25} &= \sqrt{25(-1)} = 5\sqrt{-1} \end{aligned} \right\} = 8\sqrt{-1}, \text{ sum.}$$

2. Subtract $\sqrt{-4}$ from $\sqrt{-9}$.

$$\sqrt{-9} = \sqrt{9(-1)} = 3\sqrt{-1}$$

$$\sqrt{-4} = \sqrt{4(-1)} = 2\sqrt{-1}$$

$\sqrt{-1}$, difference.

EXERCISES.

Find the sum of:

1. $\sqrt{-25}$ and $\sqrt{-81}$.

2. $\sqrt{-36}$ and $\sqrt{-121}$.

3. $\sqrt{-49}$ and $\sqrt{-144}$.

4. $\sqrt{-x^2}$ and $\sqrt{-y^2}$.

5. $\sqrt{-(a+b)^2}$ and $\sqrt{-(a-b)^2}$.

Subtract:

6. $\sqrt{-16}$ from $\sqrt{-36}$.

7. $\sqrt{-25x^2}$ from $\sqrt{-49x^2}$.

8. $\sqrt{-98}$ from $\sqrt{-128}$.

9. $\sqrt{-75m^2}$ from $\sqrt{-243m^2}$.

10. $\sqrt{-a^2 + 2ab - b^2}$ from $\sqrt{-a^2 - 2ab - b^2}$.

Multiplication of Imaginaries.

1. $-4 \times (-9) = +36$, but $\sqrt{-4} \times \sqrt{-9} = \sqrt{36(-1)^2}$
 $= 6 \times (-1) = -6$.

For $\sqrt{-4} = 2\sqrt{-1}$
 and $\sqrt{-9} = 3\sqrt{-1}$ } Multiplying

we have $6(\sqrt{-1})^2 = 6(-1) = -6$.

2. Multiply $\sqrt{-a^2}$ by $\sqrt{-b^2}$, and $\sqrt{-5}$ by $\sqrt{-3}$.

Process.

$$\sqrt{-a^2} = a\sqrt{-1}$$

$$\sqrt{-b^2} = b\sqrt{-1}$$

$$\begin{aligned} ab(\sqrt{-1})^2 &= ab(-1) \\ &= -ab. \end{aligned}$$

Process.

$$\sqrt{-5} = \sqrt{5} \times \sqrt{-1}$$

$$\sqrt{-3} = \sqrt{3} \times \sqrt{-1}$$

$$\begin{aligned} \sqrt{15} \times (\sqrt{-1})^2 \\ = \sqrt{15} \times (-1) = \\ -\sqrt{15}. \end{aligned}$$

EXERCISES.

Multiply :

1. $5\sqrt{-2}$ by $3\sqrt{-2}$.
2. $\sqrt{-b^2}$ by $\sqrt{-y^2}$.
3. $-4\sqrt{-x}$ by $5\sqrt{-y}$.
4. $\sqrt{-7} \times \sqrt{-7}$ by $\sqrt{-7}$.
5. $\sqrt{-a^2} \times \sqrt{-b^2}$ by $\sqrt{c^2}$.
6. $\sqrt{-4} \times \sqrt{-5}$ by $\sqrt{-6}$.
7. $2 - 3\sqrt{-2}$ by $4 + \sqrt{-2}$.
8. $5 + \sqrt{-8}$ by $9 - 3\sqrt{-8}$.
9. $2\sqrt{-4} - 4\sqrt{-3}$ by $5\sqrt{-4} + 7\sqrt{-3}$.
10. $\sqrt{-1} \times \sqrt{-4}$ by $\sqrt{-25} \times \sqrt{-49}$.

Expand :

1. $(2\sqrt{-2})(3\sqrt{-3})$.
2. $(5 + \sqrt{-7})(5 - \sqrt{-7})$.
3. $(4\sqrt{-3} + 7\sqrt{-2})(4\sqrt{-3} - 7\sqrt{-2})$.
4. $(\sqrt{-2} - 3\sqrt{-3})^2$.
5. $(2\sqrt{3} - \sqrt{-5})(4\sqrt{3} - 2\sqrt{-5})$.

Division of Imaginaries.

$$\frac{36}{4} = 9, \text{ and } \frac{\sqrt{-36}}{\sqrt{-4}} = \sqrt{+9} = 3.$$

$$\text{For } \frac{\sqrt{-36}}{\sqrt{-4}} = \frac{6\sqrt{-1}}{2\sqrt{-1}} = \frac{6}{2} = 3.$$

$$\text{In general, } \frac{\sqrt{-a}}{\sqrt{-b}} = \sqrt{\frac{a}{b}}.$$

Divide $(2\sqrt{8} - \sqrt{-10})$ by $(-\sqrt{-2})$.

Process.

$$\frac{2\sqrt{8} - \sqrt{-10}}{-\sqrt{-2}} = -2\sqrt{-4} + \sqrt{5} = \sqrt{5} - 4\sqrt{-1}. \quad \text{Ans.}$$

EXERCISES.

Divide:

1. $\sqrt{-36}$ by $\sqrt{-9}$.

2. $\sqrt{-9}$ by $\sqrt{-2\frac{1}{4}}$.

3. $5\sqrt{-35}$ by $2\sqrt{7}$.

4. $\sqrt{3}$ by $\frac{1}{2}\sqrt{-5}$.

5. $1 - \sqrt{-2} + 4$ by $1 + \sqrt{-2}$.

6. $\sqrt[4]{-108}$ by $\sqrt[4]{-3}$.

QUADRATIC SURDS.

An irrational quantity of the second degree is called a Quadratic Surd; as $\sqrt{6}$, $\sqrt{x+y}$.

1. A Quadratic Surd cannot be equated with a quadratic surd plus a rational quantity.

Assume $\sqrt{a} = \sqrt{b} + c$.

Squaring, $a = b + 2c\sqrt{b} + c^2$.

That is, $2c\sqrt{b} = a - b - c^2$.

Whence, $\sqrt{b} = \frac{a-b-c^2}{2c}$, making a surd equal to a rational quantity, which is impossible.

2. If $\sqrt{a} + b = \sqrt{c} + d$, then $\sqrt{a} = \sqrt{c}$, and $b = d$.

Assume, not $b = d$, but $b = d + x$.

Substituting, we have $\sqrt{a} + d + x = \sqrt{c} + d$.

Whence, $\sqrt{a} + x = \sqrt{c}$, which is impossible by 1 above.

Wherefore, $b = d$, and, in consequence, $\sqrt{a} = \sqrt{c}$.

3. If (1) $\sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y}$, then

$$(2) \sqrt{a - \sqrt{b}} = \sqrt{x} - \sqrt{y}.$$

Squaring (1), $a + \sqrt{b} = x + 2\sqrt{xy} + y$.

Whence by 2, $a = x + y$, A.

and $\sqrt{b} = 2\sqrt{xy}$, B.

Subtracting B from A, $a - \sqrt{b} = x - 2\sqrt{xy} + y$.

Extracting square root, $\sqrt{a - \sqrt{b}} = \sqrt{x} - \sqrt{y}$.

Simplify $\sqrt{22 - 3\sqrt{32}}$.

Explanation.

Assume $\sqrt{22 - 3\sqrt{32}} = \sqrt{x} - \sqrt{y}$ (1)

By 3, $\sqrt{22 + 3\sqrt{32}} = \sqrt{x} + \sqrt{y}$ (2)

Multiplying (1) by (2), $\sqrt{484 - 288} = x - y$

or $x - y = 14$ (3)

Squaring (1), $22 - 3\sqrt{32} = x - 2\sqrt{xy} + y$

By 2, $x + y = 22$ (4)

Adding (3) and (4), $2x = 36$

$$x = 18$$

Substituting in (3), $y = 4$

Substituting in (1), $\sqrt{22 - 3\sqrt{32}} = \sqrt{18} - \sqrt{4} = 3\sqrt{2} - 2$. Ans.

EXERCISES.

Simplify :

1. $\sqrt{9 + 2\sqrt{8}}.$

9. $\sqrt{19 + 8\sqrt{3}}.$

2. $\sqrt{3 + 2\sqrt{2}}.$

10. $\sqrt{4 + \sqrt{15}}.$

3. $\sqrt{7 + \sqrt{48}}.$

11. $\sqrt{9 - 2\sqrt{14}}.$

4. $\sqrt{10 + 4\sqrt{6}}.$

12. $\sqrt{-5 + 12\sqrt{-1}}.$

5. $\sqrt{16 + 6\sqrt{7}}.$

13. $\sqrt{33 + 20\sqrt{2}}.$

6. $\sqrt{1 + \sqrt{+1}}.$

14. $\sqrt{2m + 2\sqrt{m^2 - n^2}}.$

7. $\sqrt{26 + 4\sqrt{30}}.$

15. $\sqrt{2m - 2\sqrt{m^2 - x^2}}.$

8. $\sqrt{75 + 12\sqrt{21}}.$

EQUATIONS.

Equations containing the unknown quantity as a radical, or as a part of a radical, may be solved by Involution.

1. Solve $\sqrt{x^2 - 3} - x = -2.$

1. Transposing, $\sqrt{x^2 - 3} = x - 2.$

2. By Involution, $x^2 - 3 = x^2 - 4x + 4.$

3. Rejecting x^2 , $-3 = -4x + 4.$

4. Transposing, $4x = 4 + 3 = 7.$

5. Dividing, $x = \frac{7}{4} = 1\frac{3}{4}.$ Ans.

2. Solve $\sqrt{x - 32} = 16 - \sqrt{x}.$

1. By Involution, $x - 32 = 256 - 32\sqrt{x} + x.$

2. Rejecting x , $-32 = 256 - 32\sqrt{x}.$

3. Dividing, $-1 = 8 - \sqrt{x}.$

4. Transposing, $\sqrt{x} = 9$

5. By Involution, $x = 81,$ Ans.

3. Solve $\frac{\sqrt{x}-3}{\sqrt{x}+7} = \frac{\sqrt{x}-4}{\sqrt{x}+1}$.

1. Clearing of fractions, $(\sqrt{x}-3)(\sqrt{x}+1) = (\sqrt{x}-4)(\sqrt{x}+7)$.

2. Performing operation indicated, $x-2\sqrt{x}-3 = x+3\sqrt{x}-28$.

3. Simplifying, $5\sqrt{x} = 25$.

4. Dividing, $\sqrt{x} = 5$.

5. By Involution, $x = 25$. Ans.

EXERCISES.

Solve :

1. $4\sqrt{x} - 8 = 8$.

2. $\sqrt{x^2-3} + x = 3$.

3. $\sqrt{x+5} - \sqrt{x-1} = 2$.

4. $\sqrt{x-9} = \sqrt{x}-1$.

5. $\sqrt[3]{4x+3} = 3$.

6. $\sqrt{2x + \sqrt{x^4 - x^2}} = x + 1$.

7. $\sqrt{x^2 - 3x + 6} = 2 - x$.

8. $\sqrt{2x+5} = \sqrt{2x} + 1$.

9. $2 + \sqrt{x+3} = \sqrt{x-2} + 3$.

10. $\frac{1}{\sqrt{x}} - \sqrt{x+2} = \sqrt{x}$.

11. $x - \frac{1}{2}\sqrt{x} = \sqrt{x^2-x}$.

12. $\sqrt{x-32} = 16 - \sqrt{x}$.

13. $\sqrt{3+x} = \frac{6}{\sqrt{3+x}}$.

14. $\sqrt{x^2-5x+10} = x-1$.

$$15. \frac{\sqrt{x-8}}{\sqrt{x-6}} = \frac{\sqrt{x-4}}{\sqrt{x+2}}.$$

$$16. \sqrt{b+x} + \sqrt{b-x} = c.$$

$$17. \sqrt{8x+17} - \sqrt{2x} = \sqrt{2x+9}.$$

$$18. \sqrt{x} + \sqrt{x-4} = \frac{4}{\sqrt{x-4}}.$$

$$19. \sqrt{9+x\sqrt{x^2-3}} = x-3.$$

$$20. a+x = \sqrt{a^2+x}\sqrt{b^2+x^2}.$$

QUADRATIC EQUATIONS.

A *Quadratic Equation* contains the second power of the unknown quantity, or both the second and the first powers of the unknown quantity, but no higher powers.

A *Pure Quadratic Equation* contains only the second power of the unknown quantity ; as, $ax^2 = m$.

A *Complete Quadratic Equation* contains both the second and first powers of the unknown quantity ; as, $ax^2 + bx + m = 0$.

The Solution of Pure Quadratic Equations.

1. Solve $3x^2 - 2 = x^2 + 6$.

1. Transposing, $3x^2 - x^2 = 6 + 2$

2. Simplifying, $2x^2 = 8$

3. Dividing, $x^2 = 4$

4. Extracting root, $x = \pm 2$. (Two roots : $+2^2 = 4$; $-2^2 = 4$.)

2. Solve $\frac{3-x^2}{11} + \frac{x^2+5}{6} = 3$.

1. Clearing, $18 - 6x^2 + 11x^2 + 55 = 198$.

2. Transposing, $-6x^2 + 11x^2 = 198 - 55 - 18$.

3. Simplifying, $5x^2 = 125$.

4. Dividing, $x^2 = 25$.

5. Extracting root, $x = \pm 5$.

EXERCISES.

Solve :

1. $x^2 - 18 = 18.$

2. $4x^2 - 8 = 28.$

3. $7x^2 + 5 = 5x^2 + 55.$

4. $(3x - 4)(3x + 4) = 65.$

5. $\frac{x^2 + 1}{5} = 10.$

6. $\frac{x^2}{8} - \frac{1}{2} + x^2 = 0.$

7. $\frac{\sqrt{x}}{x - cx} = \frac{x}{\sqrt{x}}.$

8. $\frac{5}{4 - x} = \frac{8}{3} - \frac{5}{4 + x}.$

9. $\frac{6x^2 - 10}{4} = x^2 - 1.$

10. $\frac{3}{2 - x} + \frac{3}{2 + x} = 4.$

11. $\frac{x - a}{x + a} + \frac{x + a}{x - a} = 5.$

12. $(x + 1)(x - 1) = 2x^2 - 4.$

13. $\frac{2x^2 - 6}{2} - \frac{x^2 - 4}{4} - \frac{5x^2 - 10}{7} = 0.$

14. $x\sqrt{6 + x^2} = 1 + x^2.$

15. $x\sqrt{a + x^2} = b + x^2.$

16. $x - a + \sqrt{x^2 - 2ax} = b.$

Solution of Complete Quadratic Equations.

1. Solve $x^2 + 2ax + a^2 = 4a^2$.

NOTE.—The first member is a perfect square, and the a^2 is the square of one-half the coefficient $2a$.

1. Extracting the square root, $x + a = \pm 2a$.

2. Simplifying, $x = \pm 2a - a = + a \text{ or } -3a$. Ans.

2. Solve $x^2 - 2ax = a^2$.

1. Adding the square of
one-half $2a$,

$$x^2 - 2ax + a^2 = 2a^2.$$

2. Extracting square root, $x - a = \pm a\sqrt{2}$.

3. Simplifying, $x = a \pm a\sqrt{2} = a(1 \pm \sqrt{2})$. Ans.

Hence, brief directions for solution are :

1. Reduce the equation to the form $x^2 + 2px = q$.

2. Add to both members the square of one-half the coefficient of x .

3. Extract the square root of both members.

4. Simplify the result.

EXERCISES.

1. Solve $x^2 - 7x = 8$.

1. Completing square, $x^2 - 7x + (\frac{7}{2})^2 = 8 + (\frac{7}{2})^2$.

2. Extracting square root, $x - \frac{7}{2} = \pm \sqrt{8 + (\frac{7}{2})^2}$.

3. Simplifying, $x = \frac{7}{2} \pm \sqrt{8 + (\frac{7}{2})^2} =$

$$\frac{7}{2} \pm \sqrt{\frac{32 + 49}{4}} = \frac{7}{2} \pm \frac{9}{2} = 8 \text{ or } -1.$$

Ans.

2. Solve $3x^2 - 10x = 32$.

1. Reducing, $x^2 - \frac{10}{3}x = \frac{32}{3}$.

2. Completing square, $x^2 - \frac{10}{3}x + (\frac{5}{3})^2 = \frac{32}{3} + (\frac{5}{3})^2$.

3. Extracting square root, $x - \frac{5}{3} = \pm \sqrt{\frac{32}{3} + (\frac{5}{3})^2}$.

4. Simplifying, $x = \frac{5}{3} \pm \sqrt{\frac{96}{9} + \frac{25}{9}} = \frac{5}{3} \pm \sqrt{\frac{121}{9}}$
 $= \frac{5}{3} \pm \frac{11}{3} = 5\frac{1}{3} \text{ or } -2$. Ans.

1. Solve $x^2 + 4x = 12$.

1. Completing the square, $x^2 + 4x + 4 = 16$.

2. Extracting square root, $x + 2 = \pm 4$.

3. Simplifying, $x = -2 \pm 4 = 2$ and -6 . Ans.

4. $x^2 - 8x = 20$.

5. $x^2 - 3x = 28$.

6. $x^2 + 22x = 48$.

7. $x^2 + 12x = 13$.

8. $(x + 10)^2 = 28$.

9. $x^2 - \frac{2}{3}x = 1\frac{3}{3}$.

10. $x^2 - \frac{5}{6}x = \frac{1}{6}$.

11. $5x^2 - 7 = 10x + 68$.

12. $\frac{1}{x-2} - \frac{2}{x+2} = \frac{3}{5}$.

13. $5x^2 + 14x = 55$.

14. $x + 22 - 6x^2 = 0$.

15. $15 = 17x + 4x^2$.

16. $20x^2 = 12 - x$.

17. $15x^2 - 2ax = a^2$.

18. $12x^2 - cx - 20c^2 = 0$.

19. $\frac{5x+7}{x-1} = 3x+2$.

20. $\frac{2x-1}{3} + \frac{3}{2x-1} = 2$.

21. $\frac{2x-1}{x} - \frac{3x}{3x-1} + \frac{1}{2} = 0$.

The Factorial Method.

Assume $(x - n) (x - n') = 0$.

The roots of the equation are n and n' , for when $x = n$, the factor $x - n = 0$, and the assumed equation becomes $0 \times (x - n') = 0$; when $x = n'$ the factor $x - n' = 0$, and the assumed equation becomes $(x - n) \times 0 = 0$.

1. Solve $x^2 - 7x + 10 = 0$.

1. Factoring, $(x - 2) (x - 5) = 0$.
2. Equating to 0, $x - 5 = 0$, and $x - 2 = 0$.
3. Whence we have $x = 5$, and $x = 2$.

2. Solve $6x^2 - 5x - 6 = 0$.

1. Factoring, $(3x + 2) (2x - 3) = 0$.
2. Equating to 0, $2x - 3 = 0$, and $x = \frac{3}{2}$.
 $3x + 2 = 0$, and $x = -\frac{2}{3}$.

3. Solve $x^2 + \frac{7}{3}x = 2$.

1. Clearing, $3x^2 + 7x - 6 = 0$.
2. Factoring, $(3x - 2) (x + 3) = 0$.
3. Equating to 0, $x + 3 = 0$, and $3x - 2 = 0$.
4. Transposing, $x = -3$, and $x = \frac{2}{3}$.

4. Solve $x^3 + x^2 - 4x^2 + 4 = 0$.

1. Factoring, $x^2 (x + 1) - 4 (x^2 - 1) = 0$.
2. Factoring, $(x + 1) (x^2 - 4x + 4) = 0$.
3. Factoring, $(x + 1) (x - 2) (x - 2) = 0$.
4. Equating to 0, $x + 1 = 0$, $x - 2 = 0$, $x - 2 = 0$.
5. Hence, the roots -1 , $+2$, and $+2$.

5. Solve $-x^2 + x = \frac{6}{25}$.

1. Transposing, $x^2 - x + \frac{6}{25} = 0$.
2. Factoring, $(x - \frac{3}{5}) (x - \frac{2}{5}) = 0$.
3. Equating to 0, $x - \frac{3}{5} = 0$, and $x - \frac{2}{5} = 0$.
4. Hence, the roots $= +\frac{3}{5}$ and $+\frac{2}{5}$.

6. Solve $adx - acx^2 = bcx - bd$.

1. Transposing, $acx^2 - adx + bcx - bd = 0$.

2. Dividing by ac , $x^2 - \frac{dx}{c} + \frac{bx}{a} - \frac{bd}{ac} = 0$.

3. Factoring, $x^2 - \left(\frac{d}{c} - \frac{b}{a}\right)x - \frac{bd}{ac} = 0$.

4. Factoring, $\left(x - \frac{d}{c}\right)\left(x + \frac{b}{a}\right) = 0$.

5. Hence, the roots are $\frac{d}{c}$ and $-\frac{b}{a}$.

EXERCISES.

Solve:

1. $x^2 - 6x + 8 = 0$.

6. $\frac{2}{x} + \frac{x}{2} = -\frac{5}{2}$.

2. $\frac{x}{x+60} - \frac{7}{3x-5} = 0$.

7. $x^2 - \frac{1}{30}ax - \frac{1}{30}a^2 = 0$.

3. $2x^2 - x - 3 = 0$.

8. $9x^2 - 99x - 190 = 0$.

4. $x^2 - 3 = \frac{x-3}{6}$.

9. $\frac{x-1}{x+1} - \frac{x+1}{x-1} = 1$.

5. $\frac{8x}{x+2} - 6 = \frac{20}{3x}$.

10. $x^2 - 2mx - 1 = 0$.

Solution by Formula.

Assume $ax^2 + bx + c = 0$.

Solving, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ and $\frac{-b - \sqrt{b^2 - 4ac}}{2a}$.

By this formula an equation of the assumed form may be readily solved.

1. Solve $2x^2 + 5x - 3 = 0$.

1. $a=2$, $b=5$, $c=-3$.

2. Substituting, $x = \frac{-5 \pm \sqrt{25 + 24}}{4}$.

3. Simplifying, $x = \frac{-5 + 7}{4} = \frac{2}{4} = +\frac{1}{2}$ and $x = \frac{-5 - 7}{4} = -3$.

EXERCISES.

- | | |
|--------------------------|-----------------------------------|
| 1. $4x^2 - x = 3.$ | 6. $3x^2 - 11x = -6.$ |
| 2. $3x^2 + 4x = 4.$ | 7. $4x^2 - 9x = 28.$ |
| 3. $8x^2 + 2x = 3.$ | 8. $5x^2 + 7x = 12.$ |
| 4. $7x^2 - 16x + 4 = 0.$ | 9. $5x^2 - 7x = 6.$ |
| 5. $5x^2 - 3 = 14x.$ | 10. $11x^2 - 9x = -\frac{10}{9}.$ |

MISCELLANEOUS EXERCISES.

- $25x = 6x^2 + 21.$
- $8x^2 + x = 30.$
- $12x^2 = 29x - 14.$
- $19x = 15 - 8x^2.$
- $50x^2 - 15x = 27.$
- $5x^2 = 8x + 21.$
- $21x^2 = 2ax + 3a^2.$
- $12x^2 + 23kx + 10k^2 = 0.$
- $(x - 1)^2 - (3x + 8)^2 - (2x + 5)^2 = 0.$
- $\sqrt{4x - 3} - \sqrt{x + 1} = 1.$
- $9\frac{1}{3}x^2 - 90\frac{1}{3}x + 195 = 0.$
- $3x^2 - 30x = 9(x - 12).$
- $\frac{3}{4}(x^2 - 3) = \frac{1}{8}(x - 3).$
- $4a^2x = (a^2 - b^2 + x)^2.$
- $\frac{x + c}{x - c} - \frac{x - c}{x + c} = 1.$
- $\sqrt{3x + 1} + \sqrt{2x - 1} = \sqrt{9x + 4}.$

$$17. \frac{x^2 - 5x}{x + 3} = x - 3 + \frac{1}{x}.$$

$$18. \frac{x}{x - 1} = \frac{3}{2} + \frac{x - 1}{x}.$$

$$19. \frac{x + 2}{x - 2} - \frac{x - 2}{x + 2} = \frac{5}{6}.$$

$$20. \frac{x - 6}{x - 12} - \frac{x - 12}{x - 6} = \frac{5}{6}.$$

PROBLEMS.

1. Divide the number 30 into two such parts that the product of the two parts may be equal to 125.

Solution by Factors.

Let x = one part,

then $30 - x$ = the other part.

and $x(30 - x) = 125$.

$$1. \quad 30x - x^2 = 125.$$

$$2. \quad x^2 - 30x + 125 = 0.$$

$$3. \quad (x - 25)(x - 5) = 0.$$

$$4. \quad x = 25 \text{ or } 5.$$

$$5. \quad 30 - x = 5 \text{ or } 25.$$

That is, when either number is the first part, the other is the second part.

Solution by Formula.

$$a = 1, b = -30, c = 125.$$

$$x = \frac{30 \pm \sqrt{900 - 500}}{2} = \frac{30 \pm 20}{2} = \frac{50}{2} \text{ or } \frac{10}{2} = 25 \text{ or } 5.$$

2. Divide a number m into two such parts that the product of the two parts may be equal to n .

3. Divide the number 112 into two such parts that the product of the smaller part and their difference will be 768.

Let x = the smaller part,

then $112 - x$ = the larger part,

and $112 - 2x$ = their difference.

Whence, $x(112 - 2x) = 768$

$$1. \quad 112x - 2x^2 = 768.$$

$$2. \quad 2x^2 - 112x = -768.$$

$$3. \quad x^2 - 56x + 384 = 0.$$

$$4. \quad a = 1, b = -56, c = 384.$$

$$5. \quad x = \frac{56 \pm \sqrt{3136 - 1536}}{2} = \frac{56 \pm 40}{2} = \frac{96}{2} \text{ or } \frac{16}{2}$$

= 48 or 8, lesser part.

. $112 - 48$ or $8 = 64$ or 104 , larger part.

4. What are the two parts of 20 whose product is equal to 24 times their difference?

5. Find a number whose square diminished by 119 is equal to ten times the excess of the number over 8.

6. A man is five times as old as his son, and the sum of the squares of their ages is equal to 2106. What are their ages?

7. The sum of a number and its square is nine times the next highest number. Find the number.

8. The sum of the reciprocals of two consecutive numbers is $\frac{21}{110}$. Find the two numbers.

9. The perimeter of a rectangular field is 360 rods, and its area is 7200 square rods. Find the length of the sides.

10. A man sold a horse for \$171, gaining as many per cent. as the horse cost him dollars. Find the cost of the horse.

11. If 12 be subtracted from a certain number, the difference between the remainder and the square of the remainder will be 30. Find the number.

12. One number is $\frac{3}{4}$ of another, and their difference plus their product is 196. Find the numbers.

13. A man bought a horse which he sold for \$24, and lost by the sale as much per cent. as the horse cost him. How much did he pay for the horse?

14. Find a number such that if you subtract it from 10, and multiply the remainder by the number itself, the product will be 21.

15. A man bought a number of yards of cloth for 240 cents. If he had received 3 yards less for the same sum, it would have cost him 4 cents more per yard. How many yards did he purchase?

16. *A man sold a number of yards of cloth for 240 cents. If he had received the same sum for 3 yards more, it would have brought him 4 cents less per yard. How many yards did he sell?*

NOTE.—The above changes in the statement of the problem (see problem 15) were necessarily made in order to obtain + 12 as the value of x .

—15 refers us to problem 15, as that in which + 15 is the value of x .

17. Find three consecutive numbers such that twice the product of the first and third is equal to the square of the second plus 62.

18. A man being asked how much money he had in his purse, replied, "The square root of the number of dollars taken from half the number gives a remainder of 180." How much money had he?

19. From A two persons start at the same time; one going due north, the other due east, at the rate of 6 and 8 miles per hour respectively. In how many hours will they be 100 miles apart?

20. A merchant bought a number of yards of cloth for

\$100 ; he kept 5 yards and sold the rest at \$2 per yard more than he gave, and received \$20 more than he originally spent. How many yards did he buy ?

21. A person rows 20 miles down a river and back again in 10 hours, and he finds that he can row 2 miles against the stream in the same time that he can row 3 miles with it. Required the rate of the stream, and the times of his going and returning.

22. The area of a rectangle is 1200 square feet ; if one of its dimensions is increased 3 feet and the other is decreased $1\frac{1}{2}$ feet, the area will be increased 60 square feet. Find the dimensions of the rectangle.

Write the problem suggested by the negative result.

23. The difference of two numbers is 3 ; the difference of their cubes is 513. Find the numbers.

Equations in Quadratic Form.

An equation is said to be in the Quadratic Form when it is *like* a quadratic equation ; namely, in having

1. Three terms.
2. The unknown quantity in two of the terms.
3. The exponent of the unknown quantity in one of the terms *twice as great as in the other*.

1. $x^6 + 2x^3 = 80$ is an equation of the *quadratic form*, and may be solved as if *quadratic*.

1. Completing square, $x^6 + 2x^3 + 1 = 81$.
2. Extracting square root, $x^3 + 1 = \pm 9$.
3. Transposing, $x^3 = \pm 9 - 1 = 8$ or -10 .
4. Extracting cube root, $x = 2$ or $\sqrt[3]{-10}$.

2. Solve $3x + 3\sqrt{x} = 36$.

1. Dividing, $x + \sqrt{x} = 12$.
2. Using fractional exponent, $x + x^{\frac{1}{2}} = 12$.
3. Completing square, $x + x^{\frac{1}{2}} + \frac{1}{4} = 12 + \frac{1}{4} = \frac{49}{4}$.
4. Extracting square root, $\sqrt{x} + \frac{1}{2} = \pm \frac{7}{2}$.
5. Transposing, $\sqrt{x} = -\frac{1}{2} \pm \frac{7}{2} = \frac{6}{2}$ or $-\frac{8}{2} = 3$ or -4 .
6. Squaring, $x = 9$ or 16 .

3. Solve $8x^{-\frac{3}{2}} - 11x^{-\frac{3}{4}} = -3$.

Solution by Formula.

1. $a=8, b=-11, c=-3$.
 2. $x^{-\frac{3}{4}} = \frac{11 \pm \sqrt{121-96}}{16} = \frac{11 \pm 5}{16} = 1$ or $\frac{3}{8}$.
 3. Extracting cube root, $x^{-\frac{1}{4}} = 1$ or $(\frac{3}{8})^{\frac{1}{3}}$.
 4. Involving, $x^{-1} = 1$ or $(\frac{3}{8})^{\frac{4}{3}}$.
 5. Inverting, $x = 1$ or $(\frac{8}{3})^{\frac{4}{3}} = \frac{16}{3}$ or $5\frac{1}{3}$. Ans.
4. Solve $(x^3 + 16)^{\frac{2}{3}} - 3(x^3 + 16)^{\frac{1}{3}} = -2$.

Solution by Formula.

1. $a=1, b=-3, c=-2$.
 2. $(x^3 + 16)^{\frac{1}{3}} = \frac{3 \pm \sqrt{9-8}}{2} = \frac{3 \pm 1}{2} = 2$ or 1 .
 3. Cubing, $x^3 + 16 = 8$ or 1 .
 4. Transposing, $x^3 = -8$ or -15 .
 5. Extracting cube root, $x = -2$ or $\sqrt[3]{-15}$. Ans.
5. Solve $(x-5)^3 - 3(x-5)^{\frac{3}{2}} = 40$.

Solution by Formula.

1. $a=1, b=-3, c=40$.
2. $(x-5)^{\frac{3}{2}} = \frac{3 \pm \sqrt{9+160}}{2} = \frac{3 \pm 13}{2} = 8$ or -5 .
3. Extracting cube root, $(x-5)^{\frac{1}{2}} = 2$ or $-\sqrt[3]{5}$.
4. Squaring, $x-5 = 4$ or $\sqrt[3]{25}$.
5. Transposing, $x = 9$ or $5 + \sqrt[3]{25}$. Ans.

6. Solve $x^2 + 5x - \sqrt{x^2 + 5x + 14} = 42$.

1. Adding 14, $(x^2 + 5x + 14) - (x^2 + 5x + 14)^{\frac{1}{2}} = 56$.

2. $(x^2 + 5x + 14)^{\frac{1}{2}} = \frac{1 \pm \sqrt{1 + 224}}{2} = \frac{1 \pm 15}{2} = 8 \text{ or } -7$.

3. Squaring, $x^2 + 5x + 14 = 64 \text{ or } 49$.

4. Transposing, $x^2 + 5x = 50 \text{ or } 35$.

5. By formula, $x = \frac{-5 \pm \sqrt{25 + 200 \text{ or } 140}}{2} = \frac{10}{2} \text{ or } \frac{14}{2} = 5 \text{ or}$

-10 ; also, $x = \frac{-5 \pm \sqrt{165}}{2}$.

7. Solve $x^4 + 2x^3 - 25x^2 - 26x + 120 = 0$.

1. Adding $+x^2$ and $-x^2$, $x^4 + 2x^3 + x^2 - 26x^2 - 26x = -120$.

2. Factoring, $(x^2 + x)^2 - 26(x^2 + x) = -120$.

3. Hence, $x^2 + x = \frac{26 \pm \sqrt{26^2 - 480}}{2} = \frac{26 \pm 14}{2} = \frac{40}{2} \text{ or } \frac{12}{2} = 20 \text{ or } 6$.

4. And $x = \frac{-1 \pm \sqrt{1 + 80}}{2} = \frac{8}{2} \text{ or } \frac{-10}{2} = 4 \text{ or } -5$;

Also, $x = \frac{-1 \pm \sqrt{1 + 24}}{2} = \frac{4}{2} \text{ or } \frac{-6}{2} = 2 \text{ or } -3$.

Extra.

8. Solve $x^3 + x^2 - x - 1 = 0$.

1. Factoring, $x^2(x + 1) - (x + 1) = 0$.

2. Factoring, $(x + 1)(x^2 - 1) = 0$.

3. Equating to 0, $x = -1$, and $x = \pm 1$.

EXERCISES.

1. $x^4 - 21x^2 = 100$.

2. $x^4 - 8x^2 = 9$.

3. $x^3 - x^{\frac{3}{2}} = 56$.

4. $x^6 - 65x^3 = -64$.

5. $x + x^{\frac{1}{2}} = 6.$
6. $x^{-1} - x^{-\frac{1}{2}} = 6.$
7. $5\sqrt{x+4} + (x+4) = 24.$
8. $(\frac{1}{4}x^2 + x) - 40 = 3\sqrt{\frac{1}{4}x^2 + x}.$
9. $3x^{-\frac{2}{3}} - 7x^{-\frac{1}{3}} = 6.$
10. $\left(\frac{6}{y} + y\right)^2 + \left(\frac{6}{y} + y\right) = 30.$
11. $x^2 - 2 = \frac{2}{1}\frac{3}{2}x.$
12. $ax^4 + 2x^2 = b.$
13. $2x^{-\frac{1}{2}} - 9x^{-\frac{1}{4}} = -4.$
14. $x^2 - 5x + 2\sqrt{x^2 - 5x + 3} = 12.$
15. $x^4 - 8x^3 + 10x^2 + 24x + 5 = 0.$
16. $\frac{x^2 - 6}{x} + \frac{5x}{x^2 - 6} = 6.$
17. $x^{\frac{3}{2}} - 2 = x^{\frac{3}{4}}.$
18. $x^4 - 14x^3 + 61x^2 - 84x + 20 = 0.$

Simultaneous Equations Involving Quadratics.

No rule applicable in all cases can be given for the solution of simultaneous equations involving quadratics; but the intelligent use of the principles and rules already established clears away the difficulties that arise.

1. Solve $x - y = 1.$ (1)

$xy = 12.$ (2)

1. From (1), $y = x - 1.$ (3)

2. From (2) (3), $x(x-1) = 12.$

3. Hence, $x^2 - x = 12.$

4. $x = \frac{1 \pm \sqrt{1+48}}{2} = \frac{7}{2} \text{ or } -\frac{5}{2} = 4 \text{ or } -3.$

5. From (3), $y = 4 - 1 = 3,$ or $-3 - 1 = -4.$

2. Solve $x + y = 5$. (1)

$$x^2 + y^2 = 13. \quad (2)$$

1. From (1), $x^2 + 2xy + y^2 = 25$. (3)

2. From (2) (3), $2xy = 12$. (4)

3. From (4), $xy = 6$. (5)

4. From (5), $y = \frac{6}{x}$. (6)

5. From (6) (1), $x + \frac{6}{x} = 5$.

6. Multiplying by x , $x^2 + 6 = 5x$.

7. Transposing, $x^2 - 5x = -6$.

8. Solving, $x = \frac{5 \pm \sqrt{25 - 24}}{2} = \frac{5 \pm 1}{2} = 3 \text{ or } 2$.

9. From (6), $y = \frac{6}{3 \text{ or } 2} = 2 \text{ or } 3$.

3. Solve $x^2 + y^2 = 185$. (1)

$$x + y = 17. \quad (2)$$

1. From (2), $x^2 + 2xy + y^2 = 289$. (3)

2. From (3) (1), $2xy = 104$. (4)

3. And $xy = 52$. (5)

4. From (2) (5), $x = 13 \text{ or } 4$.

And $y = 4 \text{ or } 13$.

4. Solve $x^3 - y^3 = 19$. (1)

$$x - y = 1. \quad (2)$$

1. From (1) (2), $x^2 + xy + y^2 = 19$. (3)

2. From (2), $x^2 - 2xy + y^2 = 1$. (4)

3. From (3) (4), $3xy = 18$.

And $xy = 6$. (5)

4. From (2) (5), $x = 3 \text{ or } -2$.

$y = 2 \text{ or } -3$.

5. Solve $x^2 - xy = 3$. (1)

$x + 2y = 7$. (2)

1. From (2), $x = 7 - 2y$. (3)

2. From (3) (1), $(7 - 2y)^2 - y(7 - 2y) = 3$.

3. Expanding and reducing, $6y^2 - 35y = -46$.

4. $y = \frac{35 \pm \sqrt{35^2 - 1104}}{12} = \frac{23}{6}$ or 2.

5. From (3), $x = 3$ or $-\frac{2}{3}$.

6. Solve $x^2 + 2xy = 24$. (1)

$2xy + 4y^2 = 120$. (2)

1. Let $y = vx$.

2. From (1), $x^2 + 2vx^2 = 24$. (3)

3. From (2), $2vx^2 + 4v^2x^2 = 120$. (4)

4. From (3), $x^2 = \frac{24}{1 + 2v}$.

5. From (4), $x^2 = \frac{120}{2v + 4v^2}$.

6. From (4) (5), $\frac{24}{1 + 2v} = \frac{120}{2v + 4v^2}$.

7. Or $\frac{1}{1 + 2v} = \frac{5}{2v + 4v^2}$.

8. $2v + 4v^2 = 5 + 10v$.

9. $4v^2 - 8v = 5$.

10. $v = \frac{8 \pm \sqrt{64 + 80}}{8} = \frac{8 \pm 12}{8} = \frac{20}{8}$ or $-\frac{4}{8} = \frac{5}{2}$ or $-\frac{1}{2}$.

11. From (5), $x^2 = \frac{24}{1 + 5} = \frac{24}{6} = 4$.

And $x = \pm 2$.

12. From (1), $y = 2x(\pm 2) = 4$ or -4 .

The other values of x and y are readily found.

7. Solve $x^4 + y^4 = 82$. (1)

$x - y = 2$. (2)

1. $x = u + v$ and $y = u - v$. (3)

2. From (2), $v = 1$.

3. From (1), $(u + 1)^4 + (u - 1)^4 = 82$.

4. $\therefore 2(u^4 + 6u^2 + 1) = 82$.

5. $u^4 + 6u^2 - 40 = 0$.

6. $u^2 = 4$ or -10 .

7. And $u = \pm 2$, or $\pm \sqrt{-10}$.

8. $x = u + v = 4, -1, 1 \pm \sqrt{-10}$.

9. $y = u - v = 1, -3, -1 \pm \sqrt{-10}$.

8. Solve $\frac{1}{x} + \frac{1}{y} = 2$. (1)

$x + y = 2$. (2)

1. From (2), $y = 2 - x$. (3)

2. From (2) (1), $\frac{1}{x} + \frac{1}{2-x} = 2$.

3. Clearing, $2 - x + x = 4x - 2x^2$.

4. $2x^2 - 4x = -2$.

7. $(x - 1)(x - 1) = 0$.

5. $x^2 - 2x = -1$.

8. $x = 1$.

6. $x^2 - 2x + 1 = 0$.

9. $y = 2 - 1 = 1$.

EXERCISES.

Solve :

1. $x + y = 5$.

$x^2 - 2y^2 = 1$.

4. $2x^2 - xy = 6y$.

$x + 2y = 7$.

2. $x + y = 9$.

$xy = 20$.

5. $x + y = 2$.

$xy = -15$.

3. $x - y = 2$.

$xy = 15$.

6. $x^2 + y^2 = 50$.

$x - y = 8$.

7. $x^3 + y^3 = 133.$
 $x^2 - xy + y^2 = 19.$
8. $x + y = 10.$
 $x^2 + y^2 = 58.$
9. $x + y = 5.$
 $x^3 + y^3 = 35.$
10. $x - y = 8.$
 $x^3 - y^3 = 728.$
11. $2x^2 - xy = 6.$
 $2y^2 + 3xy = 8.$
12. $2x - y = 1.$
 $x^2 + 2y^2 = 22.$
13. $x^2 - xy = 6.$
 $x^2 + xy = 66.$
14. $\frac{xy}{\sqrt{\frac{x}{y}}} = 48.$
 $\frac{xy}{\sqrt{x}} = 24.$
15. $\frac{x}{y} + \frac{y}{x} = 2\frac{1}{2}.$
 $x + y = 6.$
27. $x + 4y = 14.$
 $4x - 2y + y^2 = 11.$
28. $6x - 5y = 9.$
 $x^2 + 2y = 12 + 2\sqrt{x^2 + 2y + 3}.$
29. $x + 2y = 9.$
 $3y^2 - 5x^2 = 43.$
16. $2x - 4y = 2.$
 $2x^2 - xy - 3y^2 = 112.$
17. $x + y = 1.$
 $x^2 + y^2 = 25.$
18. $x - y = 1.$
 $x^3 - y^3 = 19.$
19. $xy = a^2.$
 $x - y = 0.$
20. $xy + x^2 = 14.$
 $y^2 + xy = 35.$
21. $x^2 + y^2 = 25.$
 $x - y = -1.$
22. $x - y = c.$
 $xy = d.$
23. $x + y = 10.$
 $x^2 + y^2 = 52.$
24. $x + y = 13.$
 $\sqrt{x} + \sqrt{y} = 5.$
25. $x^4 - y^4 = m.$
 $x^2 - y^2 = n.$
26. $3x^2 + 4xy = 20.$
 $2y^2 + 5xy = 12.$

$$30. 2x - 3 = y.$$

$$10 - 2x^2 = xy.$$

$$31. 3x^2 + 165 = 16xy.$$

$$7xy + 3y^2 = 132.$$

$$32. 3x^2 + xy + y^2 = 15.$$

$$31xy - 3x^2 - 5y^2 = 45.$$

$$33. x^2 + y^2 - 9 = 3xy.$$

$$2x^2 - 6 + y^2 = 12.$$

$$34. x^4 + y^4 = 706.$$

$$x + y = 8.$$

PROBLEMS.

1. Find two numbers whose product is 54, and 5 times the first plus ten times the second equals 105.

2. Find two numbers whose sum is 20, and whose product is p .

3. The sum of two numbers equals 9, and the sum of their cubes is 189. Find the numbers.

4. The sum of two numbers is a , and the sum of their cubes is c . Find the numbers.

5. What two numbers are those whose sum multiplied by the greater is equal to 77; and whose difference, multiplied by the less, is equal to 12?

6. Divide 100 into two parts, such that the sum of their square roots is 14.

7. It is required to divide 24 into two such parts that their product shall be equal to 35 times their difference.

8. Find two numbers whose product is 255, and the sum of whose squares is 514.

9. A. is 4 years older than B. ; and the sum of the squares of their ages is 976. What are their ages?

10. Ten hundred sixty-six men are arranged in two squares, one of which has four men more on a side than the other. Find the number of men on a side of each of the squares.

11. The area of a rectangular field whose perimeter is 88 rods is 363 square rods. Find the length and breadth.

12. The numerator of a fraction + its denominator = 8. Increase each of the terms by 1, and the product of the resulting fraction and original fraction is $\frac{2}{5}$. Find the original fraction.

13. The crew of a ship were $\frac{2}{19}$ of the number of the passengers, and there were 425 more passengers than members of the crew. Find the number of the crew.

14. Some boatmen row down a river $3\frac{1}{2}$ miles and back again in $1\frac{2}{3}$ hours. If the river has a current of 2 miles an hour, find the rate at which the crew would row in still water.

15. In running a mile the forewheel of a carriage makes 132 more turns than the hind wheel ; but it would make only 88 more turns if the circumference of each wheel were increased 2 feet. Find the circumferences of the wheels.

16. Five times the square of a number + 100 equals 10 times the number + 500.

17. Two men can do a piece of work in 6 days ; but if it takes one 5 days longer than the other to do the work alone, how long will it take each to do the work alone?

18. The sum of two fractions is equal to their product, and the difference of their squares is $\frac{5}{6}$ of their product. Find the fractions.

19. Find three numbers such that their sum shall be 14, the sum of their squares 84, and the product of the first and third equal to the square of the second.

20. A farmer paid x dollars apiece for some horses, and sold them at \$75 each, and realized a gain of x per cent. on the money expended. Find the value of x .

21. The sum of the ages of a father and his son is 100 years ; also $\frac{1}{10}$ the product of their ages exceeds the father's age by 180. How old are they?

When the father is 30, can the son be 70?

22. Form the sum of the squares of the two digits that compose a number, and you have 58. From the number subtract 12 times the right-hand digit, and you have the digits reversed. Find the number.

23. The difference of the contents of two cubical boxes is 342 cubic inches, and the difference of their edges is 6 inches. Find the length of each box.

24. A. and B. sold the same kind of cloth, and together received \$35, but A. sold 3 yards less than B. Had A. sold B.'s cloth, he would have received \$24 for it. Had B. sold A.'s cloth, he would have received \$12 $\frac{1}{2}$ for it. How many yards did each sell?

Discussion of Quadratic Equations.

As we have already seen, every complete (affected) quadratic equation can be reduced to the form $ax^2 + bx + c = 0$, of which the roots are
$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The character of the two roots depends on the quality and value of $b^2 - 4ac$.

1. Assume $b^2 - 4ac = 0$.

Then $\sqrt{b^2 - 4ac} = 0$.

The roots $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ become $-\frac{b}{2a}$, and both are *real and equal*.

2. Assume $b^2 - 4ac =$ a positive quantity.

Then $\sqrt{b^2 - 4ac} =$ a positive quantity.

$b^2 - 4ac$ may be, or may not be, a perfect square.

Hence, $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ become *rational or surd* roots, *unequal and real*.

3. Assume $b^2 - 4ac =$ a negative quantity.

Then $\sqrt{b^2 - 4ac}$ is an imaginary quantity, and

$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, having both a real and an imaginary part, are *conjugate imaginary* roots of unequal value.

Therefore, without solving an equation, we may determine the quality and value of its roots by determining the quality and value of $b^2 - 4ac$.

1. Take $2x^2 - 6x = -4\frac{1}{2}$.

Then $b^2 - 4ac = 36 - 36 = 0$.

Hence, the roots are real and equal.

2. Take $2x^2 - 10x = 12$.

Then $b^2 - 4ac = 100 + 96 = 196$.

Hence, the roots are rational, real, and unequal.

3. Take $6x^2 + 14x = 2$.

Then $b^2 - 4ac = 196 + 48 = 244$.

Hence, the roots are surd, real, and unequal.

4. Take $4x^2 - 6x = -8$.

Then $b^2 - 4ac = 36 - 128 = -92$.

Hence, the roots are conjugate imaginaries and unequal.

By assigning values to the letters in $b^2 - 4ac$, an equation may be formed having roots of any prescribed character.

1. In an equation having real equal roots, $b^2 = 4ac$, numerically.

If $b^2 = 144$, $b = \pm 12$, and if $-4ac = -144$, then $-ac = -36$.

Hence, $a = 4$, and $-c = -9$.

Whence the equation, $4x^2 \pm 12x = -9$.

2. In an equation having roots rational, real, and unequal, $b^2 > \text{or} < 4ac$, numerically.

If $b^2 = 144$, $b = \pm 12$.

If $4ac = 140$, $ac = 35$.

Hence, $a = 5$, and $c = 7$.

Whence the equation, $5x^2 \pm 12x = 7$.

3. In an equation having roots surd, real, and unequal, $b^2 > \text{or} < 4ac$, numerically.

If $b^2 = 36$, $b = \pm 6$.

If $4ac = 100$, $ac = 25$.

Hence, $a = 5$, and $b = 5$.

Whence the equation, $5x^2 \pm 12x = 5$.

4. In an equation whose roots are unequal, conjugate imaginaries $-4ac > b^2$, numerically.

If $-4ac = -600$, $-ac = -150$.

If $b^2 = 49$, $b = \pm 7$.

Hence, $a = 3$, and $-c = -50$.

Whence the equation, $3x^2 \pm 7x = -50$.

Find the character of the roots of the following equations :

- | | |
|---------------------------------|-----------------------------|
| 1. $2x^2 + 10x = -12$. | 6. $6x^2 + 7x = 3$. |
| 2. $2x^2 + 4x = +6$. | 7. $8x^2 - x = +4$. |
| 3. $3x^2 + 2x = -\frac{1}{3}$. | 8. $4x^2 = 6x = -3$. |
| 4. $2x^2 + 4x = 30$. | 9. $x^2 - 7x = -11$. |
| 5. $5x^2 - 5x = 3$. | 10. $y - 12x - 14x^2 = 0$. |

Roots in Their Relation to Coefficients.

The typical quadratic equation $ax^2 + bx + c = 0$ becomes

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Its roots,

$$1) -\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}, \text{ added, } = -\frac{2b}{2a} = -\frac{b}{a};$$

$$2) -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$$

multiplied, they $= \frac{c}{a}$.

Now, $-\frac{b}{a}$ is the coefficient of x with its sign changed, and

$\frac{c}{a}$ is the second member with its sign changed.

Hence, the principles :

1. The coefficient of x , with its sign changed, = the sum of the roots.
2. The second member, with its sign changed, = the product of the two roots.

These principles established, quadratic equations of the typical form may readily be produced from given roots.

1. Produce the equation whose roots are 2 and 3.

1. Roots, 2 and 3.
2. Sum of roots, 5 ; coefficient of x , -5 .
3. Product of roots, 6 ; second member, -6 .
4. Equation, $x^2 - 5x = -6$, or $x^2 - 5x + 6 = 0$.

EXERCISES.

- | | |
|--------------------------------|--|
| 1. 3, 4. | 6. $-\frac{7}{2}, \frac{4}{9}.$ |
| 2. 2, -3. | 7. $-\frac{7}{5}, -\frac{9}{4}.$ |
| 3. 2, $-\frac{2}{5}.$ | 8. $-\frac{15}{2}, 0.$ |
| 4. 6, $-\frac{10}{3}.$ | 9. $a + 2b, a - b.$ |
| 5. $\frac{1}{3}, \frac{3}{5}.$ | 10. $\frac{p + \sqrt{q}}{2}, \frac{p - \sqrt{q}}{2}$ |

Solution by Factoring.

1. Solve $x^2 - 5x + 6 = 0$.

Factoring, we have $(x-3)(x-2) = 0$.

Hence, we see that factoring reverses the previous process and provides a method of solution; for, placing $x-3=0$, we have $x=3$, and placing $x-2=0$, we have $x=2$.

2. Solve $6x^2 + x - 15 = 0$.

Factoring, $(2x-3)(3x+5) = 0$.

Hence, $2x-3=0$, $2x=3$, and $x=\frac{3}{2}$.

$3x+5=0$, $2x=\frac{5}{3}$, and $x=-\frac{5}{3}$.

Roots of the equation, $\frac{3}{2}, -\frac{5}{3}$.

3. Solve $x^3 - 4x^2 - 12x = 0$.

1. Factoring, $x(x^2 - 4x - 12) = 0$.

2. Factoring, $x(x+2)(x-6) = 0$.

3. Hence, $x=0$; $x+2=0$, and $x=-2$; $x-6=0$, and $x=6$.

4. Roots, $0, -2, 6$.

4. Solve $x^3 - 1 = 0$.

1. Factoring, $(x-1)(x^2 + x + 1) = 0$.

2. Hence, $x-1=0$, and $x=1$; $x^2 + x + 1=0$, and $x=$

$$\frac{-1 \pm \sqrt{-3}}{2}.$$

3. Roots, $1, \frac{-1 \pm \sqrt{-3}}{2}$.

5. Solve $x^4 - 1 = 0$.

Factoring, $(x^2 + 1)(x^2 - 1) = 0$.

Factoring, $(x^2 + 1)(x + 1)(x - 1) = 0$.

Hence the roots $x = \pm \sqrt{-1}$, $x = -1$, $x = +1$.

EXERCISES.

Solve:

1. $x^2 + 7x + 12 = 0$.

2. $x^2 + 3x - 10 = 0$.

3. $x^2 - x - 20 = 0$.

4. $2x^2 + 16x + 24 = 0$.

5. $x^3 - 27x^2 + 50x = 0$.

6. $x^3 - 14x^2 - 51x = 0$.

7. $x^3 - 8 = 0$.

8. $x^6 - 1 = 0$.

9. $x^3 + x^2 + x + 1 = 0$.

10. $12x^3 + 8x^2 - 27x - 18 = 0$.

Ratio and Proportion.

The relation which the numerator of a fraction bears to the denominator of the fraction is called **Ratio**.

The fraction $\frac{3}{4}$ expresses the ratio of 3 to 4.

The ratio of a to b is therefore expressed by the fraction $\frac{a}{b}$; also, the form $a : b$ is common. a is called the *antecedent* of the ratio, and b , the *consequent*.

Obviously, ratio cannot exist between two quantities of different kinds.

A proportion is an equation expressing the equality of two ratios. If we assume $\frac{a}{b} = \frac{c}{d}$, the expression of equality is called a proportion, and the terms a , b , c , d , are called

proportionals. The proportion may be written thus: $a:b=c:d$; or, $a:b::c:d$, and is thus read: "The ratio of a to b equals the ratio of c to d ; or, a is to b as c is to d ."

The first and last terms of a proportion are called *extremes*; the second and third terms are called *means*.

The proportion $a:b=c:d$ is also $\frac{a}{b}=\frac{c}{d}$. Reducing these fractions to a common denominator, we have $\frac{ad}{bd}=\frac{bc}{bd}$.

Since the two fractions are equal, and the denominators are equal, the numerators are equal—that is, $ad=bc$. But a and d are the extremes, and b and c are the means. Hence, we have the fundamental principle of proportion:

The product of the extremes equals the product of the Means.

No four terms are proportional unless, when arranged as means and extremes, they *conform* to the foregoing principle.

If any one of the terms of a proportion is wanting, it may readily be found as follows:

If 6, 8, and 9 are the first, second, and third terms of a proportion, what is the fourth term?

1. Making x the fourth term, $6:8=9:x$.

2. Using principle, $6x=72$.

3. Dividing, $x=12$, the fourth term.

If the second term be wanting, as in $6:x=3:7$, we have

$$3x=42$$

$x=14$, the second term.

In general, if $a:b=c:x$,

then $ax=bc$,

and $x=\frac{bc}{a}$.

Again, if $a:x=c:d$,

then $ad=cx$,

and $x=\frac{ad}{c}$.

Hence the formulæ :

$$(A) \text{ Required extreme} = \frac{\text{Product of Means.}}{\text{Given Extreme}}$$

$$(B) \text{ Required mean} = \frac{\text{Product of Extremes.}}{\text{Given Mean}}$$

When the means are equal, either mean is called a *Mean Proportional* between the first and last terms, and the last term is called a *Third Proportional* to the first and second terms.

In the proportion $a : b = b : c$, b is a mean proportional and c is a third proportional.

In the proportion $a : b = c : d$, d is a *fourth proportional*.

$$\text{Assuming,} \quad a : b = b : c.$$

$$\text{Using principle,} \quad b^2 = ac.$$

$$\text{Extracting square root,} \quad b = \sqrt{ac}. \quad \text{Hence, a second principle :}$$

A mean proportional between two quantities is equal to the square root of their product.

$$\text{Assuming,} \quad ad = bc.$$

$$\text{Dividing by } bd, \quad \frac{ad}{bd} = \frac{bc}{bd}.$$

$$\text{Cancelling } d \text{ and } b, \quad \frac{a}{b} = \frac{c}{d}.$$

$$\text{Changing form,} \quad a : b = c : d. \quad \text{Hence, a third principle :}$$

If two pairs of factors give equal products, either pair may be made the extremes of a proportion, and the other pair the means.

A proportion, as $a : b = c : d$, is subject to six important transformations :

I. *Alternation*, by which the means exchange places ; as,
 $a : c = b : d$.

Proof.

Assume 1. $a : b = c : d$.
 then 2. $ad = bc$.
 and 3. $a : c = b : d$.

II. *Inversion*, by which each antecedent and its consequent exchange places ; as, $b : a = d : c$.

Proof.

Assume 1. $a : b = c : d$.
 then 2. $ad = bc$.
 and 3. $b : a = d : c$.

III. *Composition*, by which $a : a + b = c : c + d$, or $a + b : b = c + d : d$.

Proof.

Assume 1. $a : b = c : d$.
 then 2. $ad = bc$.
 and 3. $ac + ad = ac + bc$. By adding to ac .
 also 4. $a(c + d) = c(a + b)$.
 5. $a : a + b = c : c + d$.

That $a + b : b = c + d : d$, may be proved similarly.

IV. *Division*, by which $a - b : a = c - d : c$, or $a - b : b = c - d : d$.

Proof.

Assume 1. $a : b = c : d$.
 then 2. $ad = bc$.
 and 3. $ac - ad = ac - bc$. By subtracting from ac .
 also 4. $a(c - d) = c(a - b)$.
 and 5. $a - b : a = c - d : c$.

That $(a - b) : b = (c - d) : d$, may be proved similarly.

V. *Composition and Division*, by which $a + b : a - b = c + d : c - d$.

Proof.

Assume 1. $a : b = c : d$.

then 2. $\frac{a+b}{a} = \frac{c+d}{c}$. By composition.

and 3. $\frac{a-b}{a} = \frac{c-d}{c}$. By division.

also 4. $\frac{a+b}{a-b} = \frac{c+d}{c-d}$. Dividing (2) by (3).

and 5. $a+b : a-b = c+d : c-d$.

VI. *Summation*, by which, if $a : b = c : d = e : f$, etc., then $a : b = a + c + e$ etc. $: b + d + f$, etc.—that is, any antecedent is to its consequent as the sum of all the antecedents is to the sum of all the consequents.

Proof.

Assume (1) $a : b = c : d = e : f$, etc.

then (2) $ad = bc$.

$$af = be.$$

$$ab = ba.$$

and (3) $a(b + d + f) = b(a + c + e)$.

Wherefore (4) $a : b = a + c + e : b + d + f$.

QUESTIONS.

1. What is the fundamental principle of Proportion?
2. What is the second principle of Proportion?
3. The third principle?
4. On what two principles does the proof of Alternation depend?
5. On what two the proof of Inversion?
9. What axiom is used in proving Composition? What law?
7. What in proving Division?
8. In Summation state the principles for (2) and (4); the law for (3).
9. Give the Axioms; also the Principles and rules of Common Fractions applicable in Proportion.

As the ratios of a proportion find original expression in the fractional form, the general principles governing fractions are applicable to ratios.

Observe carefully the following propositions and the proof pertaining to each.

Proposition I.

If $a : b = c : d$, and $c : f = g : h$, then $ae : bf = eg : dh$.

Proof.

$$(1) \frac{a}{b} = \frac{c}{d}, \quad (2) \frac{c}{f} = \frac{g}{h},$$

$$(3) \frac{ae}{bf} = \frac{eg}{dh}, \text{ by multiplying (1) and (2).}$$

Wherefore (4) $ae : bf = eg : dh$.

Proposition II.

If $a : b = c : d$, then $a^n : b^n = c^n : d^n$ and

$$\sqrt[n]{a} : \sqrt[n]{b} = \sqrt[n]{c} : \sqrt[n]{d}.$$

Proof.

$$(1) \frac{a}{b} = \frac{c}{d}.$$

$$(2) \frac{a^n}{b^n} = \frac{c^n}{d^n}.$$

Wherefore (3) $a^n : b^n = c^n : d^n$.

That (4) $\sqrt[n]{a} : \sqrt[n]{b} = \sqrt[n]{c} : \sqrt[n]{d}$ may be proved similarly.

Proposition III.

If $a : b = c : d$, then $ma : mb = nc : nd$.

Proof.

$$(1) \frac{a}{b} = \frac{c}{d}, \quad (2) \frac{ma}{mb} = \frac{nc}{nd}.$$

Wherefore (3) $ma : mb = nc : nd$.

That (4) $\frac{a}{m} : \frac{b}{m} = \frac{c}{n} : \frac{d}{n}$ may be proved similarly.

Proposition IV.

If $a : b = c : d$, then $ma : nb = mc : nd$.

Proof.

$$(1) \frac{a}{b} = \frac{c}{d} \quad (2) \frac{ma}{nb} = \frac{mc}{nd}$$

Wherefore (3) $ma : nb = mc : nd$.

That (4) $\frac{a}{m} : \frac{b}{n} = \frac{c}{m} : \frac{d}{n}$ may be proved similarly.

Proposition V.

If $a : b = b : c$, then $a : c = a^2 : b^2$.

Proof.

$$(1) \frac{a}{b} = \frac{b}{c} \quad (2) \frac{a}{b} \times \frac{b}{c} = \frac{a}{b} \times \frac{a}{b}$$

$$(3) \frac{a}{c} = \frac{a^2}{b^2} \quad \text{By cancelling.}$$

Wherefore (4) $a : c = a^2 : b^2$ (duplicate ratio).

Proposition VI.

If $a : b = b : c = c : d$, then $a : d = a^3 : b^3$.

$$(1) \frac{a}{b} = \frac{b}{c} = \frac{c}{d}$$

$$(2) \frac{a}{b} \times \frac{b}{c} \times \frac{c}{d} = \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} = \frac{a^3}{b^3}$$

$$(3) \frac{a}{d} = \frac{a^3}{b^3}$$

Wherefore (4) $a : d = a^3 : b^3$ (triplicate ratio).

EXERCISES.

1. In the following proportions find the value of x :

1. $x : 3 = 15 : 9$.

4. $a - 1 : a = x : x^2$.

2. $\frac{2}{3} : \frac{5}{7} = \frac{2}{15} : x$.

5. $3y : 6y^2 = 6y^2 : x$.

3. $x : 3a^2 = a^3 : ab$.

6. $1 : a = a : x^2$.

2. Find a fourth proportional to :

1. $x^3, xy, 5x^2y$.
2. 3, 5, and 42.
3. $1\frac{5}{7}, 1\frac{4}{5}, 1\frac{1}{2}$.
4. 45, 68, 90.
5. $5a - b, 5c - d, a$.
6. $(a + b)(a - b), a^2 + ab + b^2, a - b$.

3. Find a third proportional to :

1. x^2y, xy .
2. $a^3, 2a^2$.
3. $a, a + b$.
4. $x - y, x - z$.

4. Find a mean proportional between :

1. x^2, y^2 .
2. $2a^3, 8a$.
3. $12by^2, 3b^3$.
4. $27x^2y^3, 3y$.

5. Write the factors in the following equations as proportionals :

1. $ab = cd$.
2. $x^2y = y^2z^2$.
3. $a^2 = b^2$.
4. $ab^3 = a^3d$.

Solve :

$$1. 7 + x : 12 + x = 5 : 6.$$

Process.

1. $6(7 + x) = 5(12 + x)$.
2. $42 + 6x = 60 + 5x$.
3. $x = 18$. Ans.

$$2. x + 4 : x - 4 = 13 : 5.$$

$$3. 2x - 5 : 3x + 2 = x - 1 : 7x + 1.$$

$$4. x^2 - 4 : x^2 - x + 3 = x + 2 : 2x + 3.$$

$$5. \frac{1}{2} + \sqrt{a} : \frac{1}{4} - a = x : \sqrt{a} - 2a.$$

6. $\left. \begin{aligned} x:y &= x+y:42 \\ x:y &= x-y:6 \end{aligned} \right\}$
7. $x-12:y+3=2x-19:5y-13=5:14.$
8. $\left. \begin{aligned} x+y:x-y &= a+b:a-b \\ x^2+y^2 &= a^2b^2(a^2+b^2) \end{aligned} \right\}$
9. $\begin{cases} 3x^3y = 81 \\ x^3+y^3:x^3-y^3 = 14:13. \end{cases}$
10. $\frac{\sqrt{x+1} + \sqrt{x-1}}{\sqrt{x+1} - \sqrt{x-1}} = \frac{4x-1}{2}.$

Suggestion.—Use Composition and Division.

PROBLEMS.

1. If $a:b=c:d$, prove that $\frac{b}{a} = \frac{d}{c}$.
2. If $a:b=c:d$, prove that $a^2:c^2=ab:cd$.
3. If $a:b=c:d$, prove that $a+c:b+d=a^2d:b^2c$.
4. If $a:b=c:d$, prove that $b-a:b=d-c:d$.
5. If $a:b=c:d$, prove that $a^2:b^2=a^2+c^2:b^2+d^2$.
6. Find two numbers of which the greater is to the less as their sum is to 42, and as their difference is to 6.
7. Solve $x^2-x-2:x-2=4x^2+5x-6:5x-6$.
8. Find two numbers whose sum is 14, and whose product is to the sum of their squares as 10 to 29.
9. Find a mean proportional between $3\frac{6}{11}$ and $\frac{6}{13}$.
10. The product of two numbers is 96, and the difference of their cubes is to the cube of their difference as 19 to 1. Find the numbers.
11. The older of two boys is twice the age of the younger; if the age of each be increased by 20 years, their ages will be as 4 to 3. Find their ages.

Arithmetical Progression.

An Arithmetical Progression is a series of numbers that increase or decrease by a *common difference*, as 7, 10, 13, 16, 19, 22; or, 12, $10\frac{1}{2}$, 9, $7\frac{1}{2}$, 6.

The common difference in the first series is 3; in the second it is $1\frac{1}{2}$. The numbers constituting a series are called its *terms*.

Two principal cases arise: (1) *To find any term of a series*; (2) *To find the sum of the terms of a series*.

1. To Find any Term.

1. Find the 6th term of the series 7, 10, 13, etc.

The common difference is plainly 3.

The 1st term = 7

The 2d term = $7 + 3 \times 1$ or $2 - 1$.

The 3d term = $7 + 3 \times 2$ or $3 - 1$.

The 4th term = $7 + 3 \times 3$ or $4 - 1$.

The 5th term = $7 + 3 \times 4$ or $5 - 1$.

The 6th term = $7 + 3 \times 5$ or $6 - 1$.

That is, each term = the first term + common difference $\times$ by the number of the term less one.

2. Find the last term of the progression $a, a + d, a + 2d, a + 3d \dots l$, the number of terms being n .

1. The common difference is d .

2. The coefficient of d in any term is one less than the number of the term.

3. Hence $l = a + d(n - 1)$.

EXERCISES.

1. Find the 50th term of the progression 1, 4, 7, 10, 13 . . .

1. $a = 1, d = 3, n = 50$.

2. Formula, $l = a + d(n - 1)$.

3. Substituting, 50th term = $1 + 3(50 - 1) = 1 + 3 \times 49 = 148$.

Ans.

2. Find the 20th term of the progression 60, 57, 54, 51 . . .

1. Formula, $l = a + d(n - 1)$.

2. $a = 60$, $d = -3$, $n = 20$.

3. Substituting, 20th term $= 60 - 3(20 - 1) = 60 - 57 = 3$.

Ans.

3. Find l in each of the following series :

1. 5, 11, 17, ... l being the 27th term.

2. 5, 7, 9, ... l " " 20th term.

3. 7, 3, 1, ... l " " 15th term.

4. $13\frac{1}{2}$, 9, $4\frac{1}{2}$, ... l " " 13th term.

5. x , $2x$, $3x$, ... l " " 25th term.

To Find the Sum of the Terms.

1. Find the sum of the series 1, 3, 5, 7, 9.

1. The series is 1, 3, 5, 7, 9

2. Reversed is 9, 7, 5, 3, 1

3. Twice the sum is $10 + 10 + 10 + 10 + 10 = 10 \times 5$.

4. Once the sum is $\frac{10 \times 5}{2}$.

5. 10 is the sum of 1 and 9, the first and last terms ;
5 is the number of terms.

That is, the sum = one-half the sum of the first and last terms $\times$ by the number of terms.

2. Find the sum of n terms of the series $a, \overline{a + d}, \overline{a + 2d} \dots \dots l$.

1. The sum of the series $a + \overline{a + d} + \overline{a + 2d} + \overline{a + 3d} \dots \dots l$.

2. Reversed is $\overline{l + d} + \overline{l - d} + \overline{l - 2d} + \overline{l - 3d} \dots \dots a$.

3. $2S = \overline{a + l} + \overline{a + l} + \overline{a + l} + \overline{a + l} \dots \dots \overline{a + l}$
 $= (\overline{a + l}) n$.

$$4. S = \frac{a + l}{2} \times n. \quad (\text{Formula}).$$

EXERCISES.

1. Find the sum of the series when the first term is 5, the last term 50, and the number of terms 10.

$$1. a = 5, l = 50, n = 10.$$

$$2. s = \frac{a + l}{2} \times n = \frac{50 + 5}{2} \times 10 = 27\frac{1}{2} \times 10 = 275. \quad \text{Ans.}$$

2. Find the last term and sum of 9, 5, 1 to 100 terms.

$$1. \text{ The formula, } l = a + d (n - 1).$$

$$2. a = 9, d = -4, n = 100.$$

$$3. \text{ The 100th term} = 9 - 4 (100 - 1) = 9 - 396 = -387. \quad \text{Ans.}$$

$$4. \text{ The formula, } S = \frac{a + l}{2} \times n.$$

$$5. \text{ Substituting, } S = \frac{9 - 387}{2} \times 100 = \frac{-378}{2} \times 100 = -18900. \quad \text{Ans.}$$

3. Find the *last term* and *sum* in the following series :

$$1. 7, 16, 25, ... \text{ to } 20 \text{ terms.}$$

$$2. 3\frac{1}{2}, 1, -1\frac{1}{2}, ... \text{ to } 19 \text{ terms.}$$

$$3. 5, 9, 13, ... \text{ to } 19 \text{ terms.}$$

$$4. 12, 9, 6, ... \text{ to } 23 \text{ terms.}$$

$$5. 4, 5\frac{1}{4}, 6\frac{1}{2}, ... \text{ to } 37 \text{ terms.}$$

$$6. 2, 9, 16, 23, ... \text{ to } 100 \text{ terms.}$$

$$7. 1, 3, 5, 7, 9, ... \text{ to } 100 \text{ terms.}$$

$$8. 1, 3, 5, 7, 9, ... \text{ to } n \text{ terms.}$$

$$9. 1, 2, 3, 4, 5, ... \text{ to } n \text{ terms.}$$

$$10. a, 0, -a, ... \text{ to } a \text{ terms.}$$

From the two fundamental formulæ, $l = a + d(n - 1)$ and $S = \frac{a + l}{2} \times n$, formulæ for finding a , d , and n are readily found.

Since $l = a + d(n - 1)$,

By transposing, **1.** $a = l - d(n - 1)$.

By transposing and dividing, **2.** $n - 1 = \frac{l - a}{d}$.

By multiplying and dividing, **3.** $d = \frac{l - a}{n - 1}$.

By this last formula we can insert any required number of means between a and l .

Insert a mean between a and l .

$$\mathbf{1.} \quad d = \frac{l - a}{n - 1}$$

$$\mathbf{2.} \quad n = 3.$$

$$\mathbf{3.} \quad d = \frac{l - a}{3 - 1} = \frac{l - a}{2}.$$

$$\mathbf{4.} \quad \text{Hence, } a + \frac{l - a}{2}, \text{ or } \frac{a + l}{2} \text{ is the mean.}$$

$$\mathbf{5.} \quad \text{And } a, \frac{a + l}{2}, l \text{ is the series.}$$

That is, *the arithmetical mean between any two terms equals one-half their sum.*

$$\text{Since } S = \frac{a + l}{2} \times n.$$

$$\text{By dividing,} \quad \mathbf{1.} \quad n = \frac{2S}{a + l}.$$

$$\text{By multiplying, transp. and dividing,} \quad \mathbf{2.} \quad a = \frac{2S - ln}{n}.$$

$$\text{By multiplying, transp. and dividing,} \quad \mathbf{3.} \quad l = \frac{2S - an}{n}.$$

Hence, we see that when *any three* of the five terms, a, l, d, n, S , are given, a formula is producible for finding the remaining two terms.

The different cases that may arise are shown in the following,

TABLE.

No.	Given.	Unknown.	Values of the Unknown Quantities.
1	a, d, n	l, S	$l = a + (n - 1) d; S = \frac{1}{2}n [2a + (n - 1) d].$
2	a, d, l	n, S	$n = \frac{l - a}{d} + 1; S = \frac{(l + a)(l - a + d)}{2d}.$
3	a, d, S	n, l	$n = \frac{d - 2a \pm \sqrt{(d - 2a)^2 + 8dS}}{2d}; l = a + (n - 1) d.$
4	a, n, l	S, d	$S = \frac{1}{2}n (a + l); d = \frac{l - a}{n - 1}.$
5	a, n, S	d, l	$d = \frac{2(S - an)}{n(n - 1)}; l = \frac{2S}{n} - a.$
6	a, l, S	n, d	$n = \frac{2S}{a + l}; d = \frac{(l + a)(l - a)}{2S - (l + a)}.$
7	d, n, l	a, S	$a = l - (n - 1) d; S = \frac{1}{2}n [2l - (n - 1) d].$
8	d, n, S	a, l	$a = \frac{2S - n(n - 1) d}{2n}; l = \frac{2S + n(n - 1) d}{2n}.$
9	d, l, S	n, a	$n = \frac{2l + d \pm \sqrt{(2l + d)^2 - 8dS}}{2d}; a = l - (n - 1) d.$
10	n, l, S	a, d	$a = \frac{2S}{n} - l; d = \frac{2(nl - S)}{n(n - 1)}.$

In the above table there are four different formulæ for finding the value of each of the quantities, a, d, l, n, S , and the pupil should have little difficulty in reproducing any one of them. In fact, he should not be allowed to use a formula in the solution of an exercise or a problem until he has first explained the derivation of the formula.

EXERCISES.

1. Given d , l , and n to find a and S , with

$$d = 3, l = 76, n = 20.$$

No. 7 in the table applies.

$$1. a = l - (n - 1) d.$$

$$2. a = 76 - (20 - 1) 3.$$

$$3. a = 76 - 57 = 19.$$

$$1. S = \frac{1}{2}n [2l - (n - 1) d].$$

$$2. S = \frac{20}{2} [152 - (19) 3].$$

$$3. S = 10 [152 - 57] = 950.$$

2. Given l , d , n to find a and S , with $l = 67$, $d = 4$, $n = 17$.

3. Given a , d , n to find l and S , with $a = 5$, $d = 3$, $n = 12$.

4. Given d , l , S to find a and n , with $d = 2$, $l = 21$, $S = 120$.

5. Given a , n , S to find d and l , with $a = 3$, $n = 15$, $S = 90$.

6. Given a , l , S to find d and n , with $a = \frac{7}{12}$, $l = -\frac{5}{8}$, $S = -2\frac{1}{4}$.

PROBLEMS.

1. The greatest term is 70, the common difference 3, and the number of terms 21. What is the least term and the sum of the terms?

2. Insert 9 arithmetical means between 2 and 5.

3. Find the expression for the sum of the natural numbers 1, 2, 3, . . . from 1 to n , inclusively.

4. A laborer agreed to make an excavation on the condition that he was to receive \$2 for the first yard, \$2.50 for the second, \$3.00 for the third, and so on. If he received \$42.50 for his labor, what was the extent of the excavation?

5. Four numbers are in arithmetical progression; the sum of the least and greatest is 46, and the sum of the squares of the other two is 1060. Find the four numbers.

Suggestion.—Let $x - 3y$ represent the least number.

6. One hundred stones being placed on the ground, in a straight line, at the distance of two yards from each other, how far will a person travel who shall bring them one by one to a basket placed at two yards from the first stone?

Geometrical Progression.

A Geometrical Progression is a series of terms in which any term is equal to the product of the preceding term and a *factor which is constant throughout the series*, as 4, 8, 16, etc., whose constant factor is 2. The constant factor is called its *common ratio*.

Two principal cases arise: (1) *To find any term of a series*; (2) *To find the sum of the terms of a series*.

1. To Find any Term.

1. Find the 6th term of the series 2, 4, 8, 16, etc.

The common ratio is 2.

$$\text{The 1st term} = 2 \times 2^0.$$

$$2\text{d term} = 2 \times 2^1.$$

$$3\text{d term} = 2 \times 2^2.$$

$$4\text{th term} = 2 \times 2^3.$$

$$5\text{th term} = 2 \times 2^4.$$

$$6\text{th term} = 2 \times 2^5 = 2 \times 32 = 64.$$

That is, each term = the first term $\times$ by the common ratio raised to a power whose index is one less than the number of the term.

Find the last term of the progression $a, ar, ar^2, ar^3, \dots$, the number of terms being n .

1. The common ratio is r .
2. The exponent of r in any term is one less than the number of the term.
3. Hence, $l = ar^{n-1}$.

EXERCISES.

1. Find the 9th term of the series $1, 2, 4, \dots$

1. Formula, $l = ar^{n-1}$.
2. $a = 1, r = 2, n = 9$.
3. $l = 1 \times 2^{9-1} = 1 \times 2^8 = 1 \times 256 = 256$.

2. Find the 7th term of the series $3, 2, \frac{4}{3}, \dots$

1. Formula, $l = ar^{n-1}$.
2. $a = 3, r = \frac{2}{3}, n = 7$.
3. $l = 3 \times (\frac{2}{3})^{7-1} = 3 \times (\frac{2}{3})^6 = 3 \times \frac{64}{729} = \frac{192}{729} = \frac{64}{243}$.

NOTE.—The common ratio may be found by dividing the second term by the first.

Find l in each of the following series :

1. $3, 6, 12, \dots, l$ being the 5th term.
2. $256, 128, 64, \dots, l$ “ “ 10th term
3. $64, 32, 16, \dots, l$ “ “ 7th term.
4. $x, x^2, x^3, \dots, l$ “ “ p th term.
5. $\frac{1}{3}, \frac{1}{2}, \frac{3}{4}, \dots, l$ “ “ 8th term.

To Find the Sum of the Terms.

1. Find the sum of the series $2, 6, 18, 54, 162$.

1. The ratio is 3.
2. Once the sum is $2 + 6 + 18 + 54 + 162$.
3. Three times the sum is $6 + 18 + 54 + 162 + 486$.
4. Twice the sum is $\frac{486 - 2}{3 - 1}$. Obtained by subtracting once the sum from 3 times the sum.

That is, the sum of a geometrical series is found by multiplying the last term by the ratio, subtracting the first term from the product, and dividing the remainder by the ratio less one.

2. Find the sum of n terms of the series $a, ar, ar^2, \dots, ar^{n-2}, ar^{n-1}$.

1. Once the sum of the series is $a + ar + \dots + ar^{n-2} + ar^{n-1}$.

2. r times the sum is $ar + ar^2 + ar^3 + \dots + ar^{n-1} + ar^n$.

3. $(r - 1)$ times the sum is $ar^n - a$. Obtained by subtracting once the sum from r times the sum.

4. Once the sum, or $S = \frac{ar^n - a}{r - 1}$.

EXERCISES.

Find the last term and sum in the following series :

1. 1, 3, 9, ... to 8 terms.
2. 4, -2, 1, $-\frac{1}{2}$, ... to 10 terms.
3. 3, -6, 12, ... to 6 terms.
4. 3, $4\frac{1}{2}$, $6\frac{3}{4}$, ... to 7 terms.
5. 9, 3, 1, ... to 11 terms.
6. 2, -6, 18, ... to 8 terms.
7. 1, $1\frac{1}{2}$, $2\frac{1}{4}$, ... to 5 terms.
8. 1, 3, 9, ... to n terms.
9. 5, -10, 20, ... to 10 terms.
10. $\frac{1}{\sqrt{2}}$, -2, $\frac{8}{\sqrt{2}}$, ... to 7 terms.

From the two fundamental formulæ, $l = ar^{n-1}$ and

$S = \frac{lr - a}{r - 1}$, formulæ for finding a and r are readily found.

Since $l = ar^{n-1}$

$$1. a = \frac{l}{r^{n-1}}.$$

$$2. r^{n-1} = \frac{l}{a}.$$

$$3. r = \sqrt[n-1]{\frac{l}{a}}.$$

By this last formula, we may insert any required number of means between a and l .

Insert a mean between a and l .

$$1. r = \sqrt[n-1]{\frac{l}{a}}.$$

$$2. n = 3.$$

$$3. r = \sqrt{\frac{l}{a}}.$$

$$4. \text{Hence, } a \times \sqrt{\frac{l}{a}} \text{ or } \sqrt{\frac{a^2 l}{a}} \text{ or } \sqrt{al} \text{ is the mean.}$$

$$5. \text{And } a, \sqrt{al}, l \text{ is the series.}$$

That is, the geometrical mean between any two terms equals the square root of their product.

$$\text{Since } S = \frac{lr - a}{r - 1}.$$

$$1. Sr - S = lr - a.$$

$$2. Sr - lr = S - a.$$

$$3. r(S - l) = S - a.$$

$$4. r = \frac{S - a}{S - l}.$$

$$\text{In like manner, } l = \frac{Sr - S + a}{r}.$$

Hence we see that when *any three* of the five terms, a, l, r, m, S , are given, a formula is producible for finding the remaining two terms.

The different cases that may arise are shown in the following

TABLE.

No.	Given	Unknown.	Values of the Unknown Quantities.
1	a, r, n	l, S	$l = ar^{n-1}; S = \frac{ar^n - a}{r - 1}.$
2	l, r, n	a, S	$a = \frac{l}{r^{n-1}}; S = \frac{lr^n - l}{r^n - r^{n-1}}.$
3	r, n, S	a, l	$a = \frac{(r-1)S}{r^n - 1}; l = \frac{(r-1)Sr^{n-1}}{r^n - 1}.$
4	a, n, l	r, S	$r = \sqrt[n-1]{\frac{l}{a}}; S = \frac{\sqrt[n-1]{l^n} - \sqrt[n-1]{a^n}}{\sqrt[n-1]{l} - \sqrt[n-1]{a}}.$
5	a, n, S	r, l	$r^n - \frac{S}{a}r = \frac{a-S}{a}; l(S-l)^{n-1} = a(S-a)^{n-1}.$
6	l, n, S	r, a	$r^n - \frac{S}{S-l}r^{n-1} = \frac{l}{l-S}; a(S-a)^{n-1} = l(S-l)^{n-1}.$
7	a, r, l	S, n	$S = \frac{lr-a}{r-1}; n = \frac{\log. l - \log. a}{\log. r} + 1.$
8	a, l, S	r, n	$r = \frac{S-a}{S-l}; n = \frac{\log. l - \log. a}{\log. (S-a) - \log. (S-l)}.$
9	a, r, S	l, n	$l = \frac{a + (r-1)S}{r}; n = \frac{\log. [a + (r-1)S] - \log. a}{\log. r}.$
10	l, r, S	a, n	$a = lr - (r-1)S; n = \frac{\log. l - \log. [rl - (r-1)S]}{\log. r} + 1.$

NOTE.—Nos. 7, 8, 9, and 10 require a knowledge of logarithms to obtain the value of n .

EXERCISES.

1. Given a, n, l to find r, S , putting $a = 1, n = 4, l = 64$.

No. 4 in the table applies.

$$1. r = \sqrt[n-1]{\frac{l}{a}}.$$

$$2. a = 1, n = 4, l = 64.$$

$$3. r = \sqrt[3]{\frac{64}{1}} = \frac{4}{1} = 4.$$

$$1. S = \frac{\sqrt[n-1]{l^n} - \sqrt[n-1]{a^n}}{\sqrt[n-1]{l} - \sqrt[n-1]{a}}.$$

$$2. S = \frac{\sqrt[3]{64^4} - \sqrt[3]{1^4}}{\sqrt[3]{64} - \sqrt[3]{1}} = \frac{256 - 1}{4 - 1} = \frac{255}{3} = 85.$$

2. Given a, r, n , to find l and S , with $a=4, r=2, n=7$.

3. Given l, r, n , to find a and S , with $l=128, r=2, n=7$.

4. Given a, r, l , to find n and S , with $a=1, r=3, l=81$.

5. Given a, r, n , to find l and S , with $a=1, r=3, n=p$.

6. Given a, n, S , to find l and r , with $a=100, n=3, S=1300$.

7. Given a, r, n , to find l and S , with $a=1, r=\frac{1}{2}, n=8$.

$$1. a = 1, r = \frac{1}{2}, n = 8.$$

$$2. l = ar^{n-1} = (\frac{1}{2})^7 = \frac{1}{128}.$$

$$1. S = \frac{a - ar^n}{1 - r} = \frac{1 - \frac{1}{256}}{1 - \frac{1}{2}} = \frac{\frac{255}{256}}{\frac{1}{2}} = \frac{255}{128} = 1\frac{127}{128}.$$

8. Given $3, -2, \frac{4}{3}$, to find l and S , with $n=6$.

9. Given $-4, \frac{4}{3}, -\frac{4}{9}$, to find l and S , with $n=7$.

10. Given a, r, n , to find l and S , with $n = \text{Infinity}$.

$$1. l = ar^{n-1}.$$

2. r being less than 1, and n being infinitely great, r^{n-1} becomes infinitely small, or equal to 0.

$$3. ar^{n-1} = a \times 0 = 0.$$

Hence 4. $l = ar^{n-1} = 0$.

$$1. S = \frac{a - ar^n}{1 - r}. \text{ (formula for decreasing series.)}$$

$$2. S = \frac{a - 0}{1 - r} = \frac{a}{1 - r}, \text{ the formula for finding the sum of an infinite series.}$$

11. Find the sum to infinity of:

1. $1, \frac{1}{4}, \frac{1}{16}, \dots$

Process.

$$S = \frac{a}{1-r} = \frac{1}{1-\frac{1}{4}} = \frac{1}{\frac{3}{4}} = \frac{4}{3}.$$

2. $60, 15, \dots$

3. $1, \frac{1}{3}, \frac{1}{9}, \dots$

4. $.9, .03, .001, \dots$

5. $.8, -.4, .2, \dots$

6. $.407$ or $.407407 \dots$

Process.

1. $.407407 \dots = \frac{407}{1000} + \frac{407}{1000000} \dots$, and $r = \frac{1}{1000}$.

2. $S = \frac{a}{1-r} = \frac{\frac{407}{1000}}{1-\frac{1}{1000}} = \frac{407}{1000} \times \frac{1000}{999} = \frac{407}{999}$. Ans.

PROBLEMS.

1. Find the geometrical mean between 3 and 48.

1. $x = \sqrt{ab} = \sqrt{3 \times 48} = \sqrt{144} = 12$.

2. The series is 3, 12, 48.

2. Find the mean between 5 and 405.

3. Find the mean between $\frac{14}{3}$ and $\frac{56}{27}$.

4. Insert three means between 1 and 81.

1. Formula 4, $r = \sqrt[n-1]{\frac{b}{a}}$.

2. $r = \sqrt[4]{\frac{81}{1}} = \sqrt[4]{81} = \frac{3}{1} = 3$.

3. Means, 1×3 or $3, 3 \times 3$ or $9, 9 \times 3$ or 27 .

4. Series, 1, 3, 9, 27, 81.

5. Insert four means between 160 and 5.

1. Formula 4, $r = \sqrt[n-1]{\frac{b}{a}}$

2. $r = \sqrt[5]{\frac{5}{160}} = \sqrt[5]{\frac{1}{32}} = \frac{1}{2}$.

3. Means, $160 \times \frac{1}{2} = 80$, $80 \times \frac{1}{2} = 40$, $40 \times \frac{1}{2} = 20$, $20 \times \frac{1}{2} = 10$.
4. Series, 160, 80, 40, 20, 10, 5.
6. Insert three means between 486 and 6.
7. Insert six means between 56 and $-\frac{7}{16}$.
8. Insert two means between x^3 and y^6 .
9. Find the series in which the sixth term is 20, and the tenth term is 320.
10. The fifth term of a series is 48, and the eighth term is -384 . What is the first term?
11. If 6 is the fourth term of a series, and 1536 the twelfth term, what is the sum of the first eleven terms?
12. Find three numbers in geometrical progression whose sum is 14 and the sum of whose squares is 84.
13. Find the sum of $\frac{3}{4}, \frac{1}{2}$ to infinity.
14. The sum of an infinite series is $\frac{2}{3}$, and the sum of the first two terms is $\frac{14}{5}$. Find the series.
15. Find the geometrical mean between $\sqrt[3]{a^2x}$ and $\sqrt[3]{ax^2}$.
16. Find the geometrical series whose sum to infinity is $4\frac{1}{2}$, and whose second term is -2 .
17. What debt may be discharged in twelve months by paying \$1 the first month, \$2 the second month, \$4 the third month, and so on, each succeeding payment being double the last; and what will be the last payment?
18. A man bought 10 bushels of wheat on condition that he should pay 1 cent for the first bushel, 3 for the second, 9 for the third, and so on to the last: what did he pay for the last bushel, and for the ten bushels?

Binomial Theorem.

The following identical equations are easily obtainable by actual multiplication :

$$(a + x)^2 = a^2 + 2ax + x^2.$$

$$(a + x)^3 = a^3 + 3a^2x + 3ax^2 + x^3.$$

$$(a + x)^4 = a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4.$$

The law of formation of exponents and coefficients is very obvious, and may be expressed thus :

$$(a + x)^4 = a^4 + \frac{4}{1} a^{4-1}x + \frac{4 \cdot 3}{1 \cdot 2} a^{4-2}x^2 + \frac{4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3} a^{4-3}x^3 + \frac{4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4} a^{4-4}x^4 (= a^0x^4 = x^4).$$

Letting n represent any positive integer, and substituting it for exponent 4, we have

$$(a + x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{1 \cdot 2} a^{n-2}x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} a^{n-3}x^3 \dots$$

If we now multiply this last equation by $(a + x)$, we shall find that $n + 1$ will have taken throughout the place of n . Hence, we conclude by induction that if the theorem be true for any value of n , it is true for the next higher value of n . Now, since the theorem is true when $n = 4$, it is true when $n = 5$, and so on continuously.

If we examine attentively the expansion

$$(a + x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{1 \cdot 2} a^{n-2}x^2 \dots \dots \dots$$

we shall find four distinct features patent in every term :

1. The sum of the exponents is n .
2. The exponent of x is one less than the number of the term.
3. The numerator and denominator of the coefficient have the same number of factors, the factors of the numerator beginning with n and dimin-

ishing by 1; the factors of the denominator beginning with 1 and increasing by 1.

4. The last factor of the denominator is the same as the exponent of x .

EXERCISES.

Expand :

1. $(a + b)^5$.

6. $\left(x - \frac{3}{y}\right)^6$.

2. $(x + b)^6$.

7. $(x + a)^7$.

3. $(a - x)^7$.

8. $\left(2a + \frac{b}{2}\right)^4$.

4. $(a + 4)^8$.

9. $(2x - 3y)^4$.

5. $(x - 2y)^5$.

10. $\left(xy + \frac{a}{y}\right)^9$.

Assuming $\left(xy + \frac{a}{y}\right)^9 = (m + n)^9$, we have $(m + n)^9 = m^9 + 9m^8n + 36m^7n^2 + 84m^6n^3 + 126m^5n^4 + 126m^4n^5 + 84m^3n^6 + 36m^2n^7 + 9mn^8 + n^9$.

Restoring, we have $x^9y^9 + 9x^8y^8 \cdot \frac{a}{y} + 36x^7y^7 \cdot \frac{a^2}{y^2}$, etc. $= x^9y^9 + 9x^8y^7a + 36x^7y^5a^2$, etc.

Write and reduce :

1. The fifth term of $(a + y)^{12}$.

1. Literal part, a^8y^4 .

2. Last factor of denominator, 4.

3. Denominator, 1 . 2 . 3 . 4.

4. Numerator, 12 . 11 . 10 . 9.

5. Fifth term, $\frac{12 \cdot 11 \cdot 10 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 4} a^8y^4 = 495a^8y^4$.

2. The sixth term of $(x - 3y)^8$.

3. The eighth term of $\left(1 - \frac{1}{a}\right)^{11}$.
4. The seventh term of $(3 - b)^9$.
5. The fourth term of $\left(4a + \frac{x}{2}\right)^{10}$.
6. The ninth term of $\left(2x^2 - \frac{1}{x}\right)^{12}$.

Why is the ninth term the only one independent of x ?

General Review.

1. Simplify :

$$1. \left(\frac{a + 2b}{2a + b} - \frac{a - 2b}{2a - b} \right) \left(\frac{1}{b} + \frac{b}{2ab} \right).$$

$$2. \left(\frac{\frac{x^2 + y^2}{y} - x}{\frac{1}{y} - \frac{1}{x}} \right) \left(\frac{x^2 - y^2}{x^3 + y^3} \right).$$

$$3. a - [5b - \{a - (5c - \overline{2c - b} - 4b) + 2a\}].$$

$$4. a - \{2b + 1 - (c + 2a) + 3 - [b + 4 - (3a - \overline{b + c})]\}.$$

$$5. \frac{a}{a^2 - b^2} \times \frac{a - b - 2b\left(\frac{a - b}{a + b}\right)}{\frac{a - b}{b} \div \frac{a + b}{a}}.$$

$$6. a - [2b + 3a - (3b - 2a - \overline{a + b} + 2a) - b - 3a].$$

$$7. \left(\frac{x}{1 + x} + \frac{1 - x}{x} \right) \div \left(\frac{x}{1 + x} - \frac{1 - x}{x} \right).$$

$$8. 3x - [2y - (x + y) - \{-y - (\overline{y - x - y})\}].$$

$$9. \left(\frac{a^2 - ax}{a^2 - x^2} \div \frac{a^2 - 2ax + x^2}{a^3 - x^3} \right) \left(1 - \frac{x^2}{a^2} \right).$$

$$10. \left(\frac{\frac{a^2 + b^2}{b} - a}{\frac{1}{a} - \frac{1}{b}} \right) \left(\frac{a^2 - b^2}{a^3 + b^3} \right).$$

$$11. x - [5y - \{x - (5z - 2z - y) + 2x - (x - 2y - z)\}].$$

$$12. \frac{x^2 + xy + y^2}{x + \sqrt{xy} + y} \div \frac{x - \sqrt{xy} + y}{x - \left(-\frac{1}{x}\right)}.$$

$$13. a - \{-2a - [-3a - (-4a - \overline{5a - 6a})]\}.$$

$$14. \frac{\frac{3a + b}{2a - b} - \frac{3a - b}{2a + b}}{\frac{2a}{4a^2 - b^2}}.$$

$$15. x - [y - 2(x - y - 2z) + 2x - 3y] - z.$$

$$16. \frac{\frac{a^4 - 2a^2b^2 + b^4}{a^3b + ab^3} \div \frac{a^2 - b^2}{a^2 + b^2}}{\frac{a^2 - b^2}{4a^2b^2}}.$$

$$17. 3a - \{5a - [4a - (b - a)] - b\} - (-a - 3b).$$

$$18. \left(\frac{2}{x} - \frac{1}{a + x} + \frac{1}{a - x} \right) \div \left(\frac{a + x}{a - x} - \frac{a - x}{a + x} \right).$$

$$19. \frac{\frac{ab - b^2}{a^2 - ab} + \frac{a^2 - b^2}{ab}}{\frac{a(a + b)}{b(a - b)}} \div \frac{1}{\frac{(a + b)^2}{a^2 - b^2}}.$$

$$20. x + y - \{-2y + 3x + (3y - x + y - x)\}.$$

Factor :

$$21. 8a^3 - b^3, a^4 + a^2b^2 + b^4, 6m^2 + 5mn - 6n^2.$$

$$22. 9x^2 - \frac{4}{9}, ab + bc - b^2 - ac.$$

23. $2ac - 3bd - 6bc + ad$, $n^2 + 2n - 80$, $x^6 - 64$,
 $a^5 + 1$, $(a - b)^2 - c^2$.
24. $6a^2 + 10ab - 4b^2$, $x^3 - 125$, $1 - x - x^2 + x^3$,
 $x^8 + x^4 + 1$.
25. $16 + 4x^2 + x^4$, $27a^3 - b^3$, $a^4 - b^4$, $x^{3m} + y^{3n}$, $2x^2 +$
 $3xy - 2y^2$.
26. $x^9 - y^9$, $x^8 + x^4y^4 + y^8$, $2a^2 - 10a - 28$, $x^5 - y^5$,
 $6a^2 + 7ax - 3x^2$.
27. $3x^2 - 6x - 105$, $1 + m^2 + m^4$, $64 - b^6$, $a^{3m} - b^{3n}$,
 $2x - xy + 2y - y^2$.
28. $3x^2 + xy - 2y^2$, $1 + x^{12}$, $a^9 - 1$, $x^2 + 4ax + 4a^2 - b^2$,
 $x^{m+1} + x^my + xy^m + y^{m+1}$.
29. $55a^2 - a - 2$, $256 - x^8$, $a^4 - 3a^2 + 1$, $x^6 + y^9$,
 $m^3 - m^2 - m + 1$.
30. $x^4 - y^8$, $a^3 - 3a^2 - 10a$, $x^2 - 2x + 1 - y^2$, $a^9 + b^6$,
 $x^8 + x^4 + 1$.

Solve :

31. $\frac{x+6}{11} - \frac{2x-18}{3} + \frac{2x+3}{4} = 5\frac{1}{3} + \frac{3x+4}{12}$.
32. $\frac{1}{x} + \frac{2a}{y} = \frac{3b+4a^2}{6ab}$, $\frac{b}{x} - \frac{1}{y} = \frac{3b^2-2a}{6ab}$.
33. $\frac{a}{x} + \frac{b}{y} = c$, $\frac{a}{y} - \frac{b}{x} = d$.
34. $\frac{1}{x} - \frac{1}{y} = a$, $\frac{1}{y} + \frac{1}{z} = b$, $\frac{1}{z} - \frac{1}{x} = c$.
35. $cx + by = 3bc$, $2bx + cy = 2(b^2 + c^2)$.
36. $abx + cdy = m$, $ay - cx = n$.
37. $\frac{a}{x} - \frac{b}{y} = c$, $\frac{c}{x} + \frac{1}{y} = ab$.

$$38. \frac{a}{x} + \frac{b}{y} = \frac{c}{d}, \quad \frac{m}{x} + \frac{n}{y} = \frac{r}{s}.$$

$$39. \frac{x+y}{4} + \frac{x-y}{3} = 5, \quad 2x - 3(x - 2y) = 9.$$

$$40. \frac{2x+a}{b} - \frac{2a-x}{a} = \frac{3x-b}{b}.$$

$$41. \frac{a^2c}{b} + x^2 = (a+x)(b+x) - a(b+c).$$

$$42. \frac{1}{x} - \frac{1}{y} = a, \quad \frac{1}{x} + \frac{1}{z} = b, \quad \frac{1}{y} - \frac{1}{z} = c.$$

$$43. \frac{8}{x} - \frac{15}{y} = -1, \quad \frac{10}{x} + \frac{9}{y} = 8.$$

Find the G. C. D. of:

$$44. x^2 - 4x + 3 \text{ and } 4x^3 - 9x^2 - 15x + 18.$$

$$45. a^2 - 5a + 6 \text{ and } a^2 + 3a - 18.$$

$$46. x^3 + 5x^2 + 7x + 3 \text{ and } 2x^3 + 12x^2 + 22x + 12.$$

$$47. a^3 - 1 \text{ and } a^5 - 1.$$

$$48. 4x^4 + 2x^3 + 4x^2 + 39x - 9 \text{ and } 8x^4 + 20x^2 + 51x + 9.$$

$$49. 5x^3 + 2x^2 - 15x - 6 \text{ and } -7x^3 + 4x^2 + 21x - 12.$$

$$50. 1 + a + a^3 - a^5 \text{ and } 1 - a^4 - a^6 + a^7.$$

$$51. \frac{1}{4}a + \frac{1}{4}a^{\frac{1}{2}}b^{\frac{1}{2}} - \frac{1}{2}b \text{ and } \frac{1}{8}a - \frac{3}{8}a^{\frac{1}{2}}b^{\frac{1}{2}} + \frac{1}{4}b.$$

Simplify:

$$52. \sqrt{a^2b - 2ab^2 + b^3}, \quad 3\sqrt{\frac{2}{5}} - 2\sqrt{\frac{1}{10}}, \quad \frac{\sqrt{75} + \sqrt{48}}{\sqrt{27}}.$$

$$53. (\sqrt[3]{a^3b^2})(\sqrt[3]{a^2b^3}), \quad \sqrt{\frac{a^2b}{3}} + a\sqrt{12b} - \sqrt{3b^3}.$$

$$54. \sqrt{50x^3y^2}, \quad \sqrt[3]{54a^7b^4}, \quad \frac{2a}{3x} \left(\sqrt{\frac{27x^4}{a^2}} \right).$$

$$55. \sqrt[3]{\frac{128x^6y^2}{81z^4}} \div \sqrt{\frac{16x^3y^2}{3z^7}}.$$

$$56. \frac{1}{2a+b} \sqrt{4a^2b+4ab^2+b^3}, 2\sqrt{16a^3b^4c^5}, \frac{a}{b} \sqrt[5]{\frac{32a}{b^3}}, \\ \sqrt{a} \times \sqrt[3]{a}.$$

$$57. \sqrt{x^3-2x^2y+xy^2}, 5\sqrt[3]{\frac{3}{5}}, \sqrt[6]{8a^3b^6}, \sqrt{a^2b-a^3}, \\ (16a^4b^5-8a^6b^3)^{\frac{1}{3}}.$$

$$58. \sqrt[n]{\frac{ab}{(a+b)^{-n}}} + \sqrt[n]{\frac{(a-b)^n}{(ab)^{-1}}}, \left(\frac{\sqrt{x^3-y^3}}{\sqrt{(x-y)^3}} \right) \\ \left(\frac{\sqrt[3]{(x-y)^4}}{\sqrt{x^2+xy+y^2}} \right).$$

$$59. \sqrt[2x]{a^{3x}b^{2x}}, \sqrt[b]{m^bn^{2b}}, \frac{a}{b^n} \sqrt{\frac{b^{2n}}{a^5}}, \frac{a}{x} \sqrt[n]{\frac{x^n}{a^{-2n}}}.$$

$$60. \sqrt{\frac{1}{a}} \div \sqrt[3]{\frac{1}{b}}, (\sqrt{2+1})(\sqrt{2-1}).$$

Solve :

$$61. x^2+3xy=28, xy+4y^2=8.$$

$$62. 15x^3+2x=8.$$

$$63. x^2+xy=12 \text{ and } xy-y^2=2.$$

$$64. 3x^2-11x=70.$$

$$65. x^3+y^3=91, x^2-xy+y^2=13.$$

$$66. 4x^2 + \frac{2x}{a} = \frac{2}{a^2}.$$

$$67. 2x^2+3xy=32, 3y^2-4xy=16.$$

$$68. 8x^2+6ax=5a^2.$$

$$69. x^2+xy=15, xy-y^2=2.$$

$$70. 5x^2-12x=108.$$

$$71. x^2-y^2=8\frac{3}{4}, xy+y^2=\frac{7}{4}.$$

$$72. x^2 + 3xy = 27, 3y^2 + 2xy = 24.$$

$$73. 2xy - x^2 = 16, 3xy - y^2 = 5.$$

$$74. x - y = 3, x^2y - xy^2 = 30.$$

$$75. abx^2 - (a^2 + b^2)x + ab = 0.$$

$$76. x + y = 8xy, x^2 + y^2 = 34x^2y^2.$$

$$77. 3x^2 - 5ax = 112a^2.$$

$$78. x^2 - xy + y^2 = 37, x + y = 10.$$

Multiply :

$$79. a^{2n} + a^n b^n + b^{2n} \text{ by } a^{2n} + a^{-n} b^{-n} + b^{-2n}.$$

$$80. (a^{2n} + x^{2n}) \text{ by } (2a^{2n} - 2x^{2n}).$$

$$81. (a^{2n} + a^n x^n + x^{2n}) \text{ by } (2a^n - 2x^n).$$

$$82. (a^{3n} - a^{2n} x^n + a^n x^{2n} - x^{3n}) \text{ by } (3a^n + 3x^n).$$

Divide :

$$83. \left(\frac{x^4}{a^4} + \frac{x^2}{a^2} + 1 \right) \text{ by } \left(\frac{x^2}{a^2} - \frac{x}{a} + 1 \right).$$

$$84. 9a - 12a^{\frac{1}{2}} + 4 + 4a^{-\frac{1}{2}} + a^{-1} \text{ by } 3a^{\frac{1}{2}} - 2 - a^{-\frac{1}{2}}.$$

$$85. x^2(y+z) - y^2(x+z) + z^2(x+y) + xyz \text{ by } x - y + z.$$

$$86. x - y \text{ by } x^{\frac{1}{4}} + y^{\frac{1}{4}}.$$

Solve :

$$87. \sqrt{2x+3} - \sqrt{2x-2} = \sqrt{8x-23}.$$

$$88. \sqrt{x+a} + \sqrt{x} + \sqrt{x-a} = 0.$$

$$89. x\sqrt{x} - 2\sqrt{x} = x.$$

$$90. \sqrt{\frac{x}{c} + \frac{a}{b}} + \sqrt{\frac{x}{c} - \frac{a}{b}} = \sqrt{\frac{4x}{c} - \frac{2a}{b}}.$$

$$91. \sqrt{2x-1} + \sqrt{2x+1} = 8.$$

$$92. 2\sqrt{x} - \sqrt{4x-11} = 1.$$

$$93. \sqrt{b^2+x} - \sqrt{a^2-x} = a-b.$$

$$94. \sqrt{x+1} + \sqrt{x-2} = \sqrt{2x+3}.$$

$$95. x^{\frac{1}{2}} - a^{\frac{1}{2}} = (x-b)^{\frac{1}{2}}.$$

Find the square root of:

$$96. 16 - 32x + 24x^2 - 8x^3 + x^4.$$

$$97. a^2 + b^2 + \frac{c^2}{z^2} + 2ab - \frac{2ac}{z} - \frac{2bc}{z}.$$

$$98. 4 - 12x + 5x^2 + 26x^3 - 29x^4 - 10x^5 + 25x^6.$$

$$99. a^4 - a^3 + \frac{a^2}{4} + 4a - 2 + \frac{4}{a^2}.$$

$$100. \frac{1}{x^4} + \frac{2}{x} + 2 + x^2 + 2x^3 + x^4.$$

$$101. 4a^2 - 4a^{\frac{5}{3}} + a^{\frac{4}{3}} - 12a^{\frac{2}{3}} + 6a^{\frac{1}{3}} + 9a^{-\frac{2}{3}}.$$

Expand by the binomial theorem:

$$102. \left(2a^2 - \frac{b}{2}\right)^7 \text{ to 4 terms.}$$

$$103. \left(2a - \frac{b^2}{3}\right)^6 \text{ to 5 terms.}$$

$$104. \left(x^2 - \frac{a}{2x}\right)^8 \text{ to 5 terms.}$$

$$105. (a^2 - b^3)^8 \text{ to 4 terms.}$$

$$106. \left(2a - \frac{3b}{4}\right)^5 \text{ to 6 terms.}$$

$$107. (2a^2 - b^3)^7 \text{ to 4 terms.}$$

$$108. \left(\frac{x}{2} - 2y^2\right)^7 \text{ to 5 terms.}$$

Write and reduce:

$$109. \text{ The third term of } (x - 3y)^{10}.$$

110. The fifth term of $(3x - 4)^{20}$.

111. The twenty-first term of $(2 - x)^{22}$.

112. The middle term of $(1 + x)^8$.

PROBLEMS.

1. The per cent. of gain made by selling a horse for \$144 equals the number of dollars which the horse cost. Find the cost of the horse.

2. The area of a certain rectangle is 2400 square feet; if its length is decreased 10 feet and its breadth increased 10 feet, its area will be increased 100 square feet. Find its length and breadth.

3. Given a number of three digits such that the first digit is one-third the last and the second twice the first. If 396 is added to the number, the order of the digits is inverted. Find the number.

4. A number is expressed by two digits whose sum is 12; if the digits are interchanged, the resulting number is less than the original number by 36. Find the number.

5. The difference of two numbers is 2 and the sum of their squares 100. Find the numbers.

6. The perimeter of a rectangular lot is 220 feet, and its area is 2925 square feet. Find its length and breadth.

7. The sum of two numbers is 17, and the difference of their squares is 119. Find the numbers.

8. The perimeter of a rectangular lot is 140 feet, and its area is 1200 square feet. Find its length and breadth.

9. A sum of money at simple interest amounted in 8 months to \$496, and in 15 months to \$510. Find the sum and the rate of interest.

10. The difference of two numbers is 3; the difference of their cubes is 513. Find the numbers.

11. A father being asked the age of himself and of his son, replied: "Five years ago my age was two years more than six times that of my son, and five years hence my age will be twenty years less than four times that of my son." Find the age of each.

12. Find four roots of the following equation :

$$2x^4 + 5x^2 = 207.$$

13. Express the interest on p dollars at r per cent. for n years.

14. The sum of two numbers is 10, and the sum of their squares is 58. Find the numbers.

15. A boy is 5 years older than his sister and one-sixth as old as his father; the sum of the ages of all three is 51. Find the age of the father.

16. Find two consecutive numbers whose product is 306.

17. Find the cube root of $x^6 + 6x^5 - 40x^3 + 96x - 64$.

18. The denominator of a certain fraction exceeds the numerator by 5, and the numerator plus one-third the denominator is equal to 15. Find the fraction.

19. Find the Least Common Dividend (L. C. Dd.) of

$$2x^3 - 3x^2 - x + 1 \text{ and } x^4 - x^3 + 2x^2 - 3x - 3.$$

20. A sum of money at simple interest amounts in three years to \$460; had the principal been one-fourth greater and the rate 1 per cent. higher, it would have amounted to \$770. Find the principal and the rate.

21. A number is composed of two digits whose sum is equal to four times the tens digit; if the digits are inter-

changed, the resulting number exceeds the original number by 36. Find the number.

22. The distance between two opposite corners of a rectangular field is 17 rods, and its perimeter is 46 rods. Find the length and breadth of the field.

23. Extract the square root of $75 - 12\sqrt{21}$.

24. Prove that if four quantities are in proportion, they will be in proportion by division, and also by inversion.

25. Find the sum of the series $3, -6, 12, -24 \dots$ to 9 terms. Derive the formula for obtaining this sum.

26. Solve $x^{\frac{2}{n}} - 5x^{\frac{1}{n}} + 6 = 0$.

27. Interpret the forms $\frac{a}{0}, \frac{a}{\infty}, \frac{0}{0}$.

28. Extract the square root of $7 - \sqrt{48}$.

29. Prove that any quantity with a negative exponent is equal to the reciprocal of that quantity with an equal positive exponent.

30. Find a third proportional to $3\frac{1}{2}$ and 6.

31. Reduce $\frac{1}{\sqrt[3]{a} + b}$ to an equivalent fraction with a rational denominator.

32. Derive the formula for the sum of the terms in arithmetic progression.

33. Change the fraction $-\frac{c - (a - b)}{c + (a + b)}$ so that the sign before the dividing line will be +.

34. Simplify $\frac{1}{x + \frac{1}{1 + \frac{x+1}{3-x}}}$.

35. If a rise of 25 per cent. in the price of eggs makes a difference of 40 in the number that can be bought for 20 shillings, how much do they cost per score?

36. A. can do a piece of work in 20 days that B. can do in 12 days. A. begins the work, but after a time B. takes his place, and the whole work is finished in 14 days from the beginning. How long did A. work?

37. A piece of work can be done by A. and B. in 4 days, by A. and C. in 6 days, and by B. and C. in 12 days. In what time can it be done by A., B., and C.?

38. Divide 243 into three parts such that one-half the first, one-third the second, and one-fourth the third part shall all be equal to one another.

39. Find two numbers, one of which is three-fifths of the other, so that the difference of their squares may be equal to 16.

40. Two towns, C. and S., are 27 miles apart. Two bicyclers start at 12 M., one from C. to S. and back, the other from S. to C. and back. They meet for the second time at 3 P. M., and they were then 9 miles from C. Find when and where they met the first time.

41. The plate of a mirror is 40 by 25 inches. The area of the frame equals that of the mirror. Find the width of the frame.

42. Find the four consecutive numbers whose product is 840.

43. Find the real root of $x^{-2n} - x^{-n} = 20$.

44. Form the equation whose roots are 2, -2, 3, and 0.

45. Find the eighth term of $\left(2a^{\frac{1}{3}} + \frac{1}{2a}\right)^{11}$.

46. Solve $3x^2 + 15x - 2\sqrt{x^2 + 5x + 1} = 2$.

47. Show that in the equation $x^2 + px + q = 0$, the sum of the roots is $-p$, and their product q .

48. Insert three geometrical means between $3\frac{2}{9}$ and 18.

49. Simplify $\left[\frac{(a^{m-n})^{m+n} (a^n)^{n+p}}{(a^m)^{m-n}} \right]^{\frac{1}{n}}$.

50. What are the roots of the equation $x^3 - 6x^2 + 3x = 18$?

51. Solve $9x + 8y = 43xy$.

$$8x + 9y = 42xy.$$

52. Simplify $\frac{y^4 - 7y^3 + 8y^2 - 12y}{2y^2 - 2y - 60}$ and

$$\frac{\left(1 + \frac{1}{x}\right) \times \left(1 - \frac{1}{x}\right)^2}{x - \frac{1}{x}}$$

53. Walking $4\frac{1}{4}$ miles an hour, A. starts $1\frac{1}{2}$ hours after B., whose rate is 3 miles per hour. Find when A. will overtake B.

54. Solve $\frac{5}{6 - \frac{5}{6 - \frac{5}{6 - 5}}} = x$.

55. A well is 400 feet deep. The cost of boring was 27 cents for the first foot, and an additional cent for each succeeding foot. Find (1) the cost of boring the last foot, and (2) the cost of boring the well.

56. Simplify $\frac{2\sqrt{10}}{3\sqrt{27}} \times \frac{15\sqrt{21}}{4\sqrt{15}} \div \frac{5\sqrt{14}}{7\sqrt{48}}$.

57. Solve $(x^2 - 5x + 2)^2 = x^2 - 5x + 22$.

MISCELLANEOUS PROBLEMS AND EXERCISES.

To add interest to the following problems and exercises, the sources whence they were taken are indicated.

NEW YORK STATE.

Examination for State Certificate.

1. (a) Define algebra. (b) Give rule for algebraic subtraction. (c) Multiplication.

2. (a) Define coefficient, exponent, term, reciprocal, and surd. Give illustrations of each. (b) Why do you change the signs of the subtrahend in subtraction of polynomials?

3. (a) Explain why the term difference does not always denote a number less than the minuend. (b) Why does multiplying a negative by a positive quantity produce a negative result?

4. Extract the cube root of $a^6 - 6a^5 + 15a^4 - 20a^3 + 15a^2 - 6a + 1$.

5. Divide the number 100 into two such parts that the sum of their square roots may be equal to 14.

6. At a certain election 36,000 votes were polled, and the candidate chosen wanted but 3000 of having twice as many votes as his opponent. How many voted for each?

7. A young man, who had just come into possession of a fortune, spent $\frac{3}{8}$ of it the first year, and $\frac{4}{5}$ of the remainder

the next year, when he had \$1420 left. What was his fortune?

8. What fraction is that whose numerator being doubled and its denominator increased by 7 the value becomes $\frac{2}{3}$; but the denominator being doubled and the numerator being increased by 2, the value becomes $\frac{3}{5}$?

9. A person purchased a number of horses for \$240. If he had obtained three more for the same money, each horse would have cost him \$4 less; required the number of horses.

10. Find two numbers such that the sums of their squares may be 89, and their sum multiplied by the greater may produce 104.

11. Define (a) mathematics; (b) like quantities; (c) an equation; (d) a ratio; (e) a proportion; (f) a series.

12. Divide $6a^3 - 3a^2b - 2a + b$ by $3a^2 - 1$.

13. Simplify the expression $2m^2 + (7m^2 + 2bc - [3m^2 - bc - x] - 2x)$.

14. Add $a + x + \frac{a-x}{a} - \frac{a^2-x^2}{ax}$.

15. Multiply $\frac{(a-x)^2}{2a} \times \frac{3a-b}{a-x} \times \frac{2c}{a^2-x^2}$.

16. Find x and y (1) $\left\{ \frac{3x}{7} + \frac{2y}{3} = 5 \right\}$.
(2) $(x = y + 4)$.

17. Solve (1) $\frac{2x+3}{5} = \frac{13}{x}$.

(2) $x^2 - 3ax = 4a^2$.

18. (a) A teacher said that her school consisted of 64 scholars, and that there were three times as many in arithmetic as in algebra, and four times as many in grammar as in

arithmetic; how many were they in each study? (b) A. said to B., "Give me \$100 of your money, and I shall have twice as much as you." B. replied, "Give me \$100 of your money, and I shall have twice as much as you"; how many dollars had each?

19. (a) A market-woman bought eggs to the amount of 65 cents, some at the rate of 2 for a cent, and some at the rate of 3 for two cents. She afterward sold them all for 120 cents, thereby gaining a half cent on each egg. How many of each kind did she buy? (b) The sum of three numbers is 59; one-half the difference of the first and second is 5, and one-half the difference of the first and third is 9; required the numbers.

20. Multiply $3a^{\frac{1}{2}}, b^{\frac{2}{3}}, cx$ by $2a^{\frac{1}{3}}, b^2, c$, and divide the result by $(-2a^2b^{\frac{2}{3}}c)^2$.

21. Write the axioms.

22. Multiply $\frac{n^2 - z^2}{3m^2} \times \frac{6n^2}{n + 2}$.

23. Given $\frac{x + c}{3} - \frac{3(x + c)}{4} = \frac{1}{3}(x + c) - c$, to find x .

24. Divide a into two parts, such that the second part shall equal m times the first part plus n .

25. A. can paper a room in $\frac{1}{2}$ of a day, B. in $\frac{1}{3}$ of a day, and C. in $\frac{1}{4}$ of a day; in what time will all do it working together?

26. If two numbers differ by unity, prove that the difference of their squares equals the sum of the numbers.

27. Value of $\sqrt[n]{(a^n c^{2n} x^{3n} z^m)}$.

28. Given $x + \sqrt{(a^2 + x^2)} = \frac{2a^2}{\sqrt{(a^2 + x^2)}}$, to find x .

29. Find two numbers in the ratio of 3 to 4, such that their sum has to the sum of their squares the ratio of 7 to 50.

College Examination.

30. Reduce $\frac{\frac{a^2 - x}{2b^4}}{\frac{c + 1}{2b x^2 y^2}}$ to a simple fraction.

31. Given $ax - bx = c + dx - m$ to find x .

32. Divide 100 into two such parts that one divided by the other may give 2 as a quotient and 19 as a remainder.

33. Three persons divided a sum of money among them in such a manner that the shares of A. and B. together amounted to \$900, the shares of A. and C. together to \$800, and the shares of B. and C. to \$700; what was the share of each?

34. Two men, A. and B., commenced trade at the same time; A. had three times as much money as B., and continued in trade; A. gained \$200 and B. \$150; A. then had twice as much money as B.; how much did each have at first?

35. The sum of two numbers is 72, and the sum of their cube roots is 6; what are the numbers?

36. Express the following in the simplest form:

$$3a - (b - c) - \{a - (2b - c) + a - b - 2c\} + [a - \{a - (a - \overline{b - c})\}].$$

37. Prove the *Law of Signs* for multiplication—that is, show that like signs give plus and unlike signs minus.

38. A certain fraction becomes $\frac{1}{2}$ if 6 is added to its denominator, and it becomes $\frac{1}{4}$ if 5 is subtracted from its numerator. Find the fraction.

39. Factor and give principle involved :

$$(a) 4a^2 - 20ax + 25x^2 ;$$

$$(b) a^2b^2 - 1 ;$$

$$(c) x^2 + 8x + 15.$$

40. Find the number whose double diminished by 24 exceeds 80 by as much as the number itself is less than 100.

41. Find the square root of $4x^4 + 5x^2 - 2x - 4x^3 + 1$.

42. A man can row a skiff 6 miles an hour with the current and 3 miles an hour against it; how far can he pass down the stream and yet return to the point from which he set out in 8 hours?

43. The sum of two numbers added to the sum of their squares gives 18, and 10 times their product is 60; what are the numbers?

44. Given $ax - bx = c + dx - m$ to find x .

45. Find the value in the following: $x^4 - 34x^2 = -225$.

46. Separate into prime factors :

$$(a) x^6 - y^6 ;$$

$$(d) g^2 - 17y - 110 ;$$

$$(b) (a + b)^2 - 16 ;$$

$$(e) x^2 + 4x - 96.$$

$$(c) a^4 + 2a^2b^2 + b^4 ;$$

47. A grocer bought a certain number of eggs, part at 2 for 5 cents and the rest at 3 for 8 cents, and paid for the whole \$1.71. He sold them at 36 cents a dozen and made 27 cents. How many of each kind did he sell?

48. The sum of the digits of a number of three figures is 13. If the number be decreased by 8 and then be divided by the sum of its second and third digits, the quotient is 25; and if 99 be added to the number the digits will be reversed. Find the number.

49. When after 9 o'clock will the hands of a clock be together?

50. $1 + x\sqrt{x^2 + 12} = 1 + x$. Find the value of x .

51. A man bought a field whose length was to its breadth as 5 to 4; the price per acre was equal to the number of rods in the length of the field, and 5 times the distance around the field equalled the number of dollars it cost. Find the length and breadth of the field.

52. A man bought a quantity of meat for \$2.16. If meat were to rise 1 cent a pound, he would get 3 pounds less for the same money. How many pounds did he get?

53.
$$\left. \begin{array}{l} x^2 + y^2 - x - y = 18 \\ xy + x + y = 19 \end{array} \right\} \text{To find the values of } x \text{ and } y.$$

54. A. set out from C. toward D. at the rate of 3 miles an hour. After he had gone 28 miles, B. set out from D. toward C., and went every hour $\frac{1}{19}$ of the entire distance; and after he had travelled as many hours as he went miles in an hour, he met A. Find the distance from C. to D.

55. Three numbers are in arithmetical progression, their sum is 18 and the sum of their squares is 158. What are the numbers?

UNIVERSITY OF THE STATE OF NEW YORK.

Regents' Examination.

56. Define *numeric equation*, *homogeneous equation*, *pure equation*, *quadratic equation*, *surd*.

57. The area of a certain rectangle is 2400 square feet; if its length is decreased 10 feet and its breadth increased 10 feet, its area will be increased 100 square feet. Find its length and breadth.

58. Given a number of three digits such that the first digit is $\frac{1}{3}$ the last and the second twice the first. If 396 is

added to the number, the order of the digits is inverted. Find the number.

59. Define *factor*, *index*, *reciprocal*, *binomial*, *evolution*.

60. Expand $\left(3a^2 - \frac{b}{2}\right)^5$ by the binomial theorem and give all the work of finding the coefficients.

61. The per cent. of gain made by selling a horse for \$144 equals the number of dollars which the horse cost ; find the cost of the horse.

62. Define *geometric progression*, *identical equation*, *binomial surd*, *ratio*.

63. Extract the square root of $7 - \sqrt{48}$.

64. Prove that any quantity with a negative exponent is equal to the reciprocal of that quantity with an equal positive exponent.

65. Show by a general method that if four numbers are in proportion they will be in proportion by alternation. Find a third proportional to $3\frac{1}{2}$ and 6.

66. The three digits of a number are in arithmetic progression ; the sum of the hundreds and tens digits is equal to the units digit, and if 396 is added to the number the order of the digits is reversed. Find the number.

67. A man sold goods for \$16 and lost as much per cent. as the goods cost ; what was the cost ? Show how each result satisfies the conditions of the problem.

68. Solve $\frac{a}{x} - \frac{b}{y} = c$, $\frac{c}{x} + \frac{1}{y} = ab$.

69. Solve $5x^2 - 12x = 108$, and prove correctness of the result.

70. Solve $2x - 1 - \frac{2x - 2}{2} = \frac{3x + 1}{5} + \frac{x + 1}{4}$.

71. Define *axiom*, *terms of expression*, *transposition*, *pure quadratic equation*, *positive integer*.

72. Simplify $3x - [2y - (x + y) - \{-y - (y - \overline{x - y})\}]$.

73. Simplify $\left(\frac{a^2 - ax}{a^2 - x^2} \div \frac{a^2 - 2ax + x^2}{a^3 - x^3}\right)\left(1 - \frac{x^2}{a^2}\right)$.

74. Solve $abx + cdy = m$.

$$ay - cx = n.$$

75. Solve $8x^2 + 6ax = 5a^2$.

76. Factor $2x^2 - xy - 10y^2$, $x^6 - y^{12}$, $1 - a^8$, $4a^{2m} - 4a^mb^n + b^{2n}$, $a^4 + a^2b^2 + b^4$.

77. Simplify $a + b\sqrt{\frac{a-b}{a+b}}$, $\frac{2b}{a}\sqrt[4]{\frac{a^4}{8b^3}}$, $5\sqrt[3]{54a^2b^3c^4}$,

$$\frac{3x}{2a}\sqrt{\frac{27a^4}{x^4}}.$$

78. Find the square root of $4 - 12x + 5x^2 + 26x^3 - 29x^4 - 10x^5 + 25x^6$.

79. Reduce to lowest terms $\frac{4x^2 + 3x - 10}{4x^3 + 7x^2 - 3x - 15}$.

80. The sum of two numbers is 17, and the difference of their squares is 119; find the numbers.

81. Define *polynomial*, *similar terms*, *homogeneous quantity*, *radical*, *surd*.

82. Simplify $a - [2b + 3a - (3b - 2a - \overline{a + b} + 2a) - b - 3a]$.

83. Simplify $\left(\frac{x}{1+x} + \frac{1-x}{x}\right) \div \left(\frac{x}{1+x} - \frac{1-x}{x}\right)$.

84. Solve $cx + by = 3bc$.

$$2bx + cy = 2(b^2 + c^2).$$

85. (a) Solve $4x^2 + \frac{2x}{a} = \frac{2}{a^2}$.

86. (b) Factor $16 + 4x^2 + x^4$, $27a^3 - b^3$, $a^4 - b^4$, $x^{3m} + y^{3n}$, $2x^2 + 3xy - 2y^2$.

87. Simplify $\frac{1}{2a+b} \sqrt{4a^2b + 4ab^2 + b^3}$, $2 \sqrt[3]{16a^3b^4c^5}$,

$$\frac{a}{b} \sqrt[5]{\frac{32a}{b^3}} \sqrt[3]{a} \times \sqrt[3]{a}.$$

88. Find the square root of $\frac{a^2}{4} + b^2 + \frac{c^2}{4} + ab - \frac{ac}{2} - bc$.

89. Expand by the binomial theorem $\left(\frac{a}{2} - \frac{2}{b}\right)^5$.

90. Reduce to its lowest terms $\frac{6x^2 - x - 1}{3x^3 + 4x^2 + 4x + 1}$.

91. The difference of two numbers is 2 and the sum of their squares is 100; find the numbers.

92. Solve $\sqrt{x+1} + \sqrt{x-2} = \sqrt{2x+3}$.

93. Solve $2x^2 + 3xy = 32$.

$$3y^2 - 4xy = 16.$$

94. The perimeter of a rectangular lot is 220 feet and its area is 2925 square feet; find its length and breadth.

95. Simplify $x - [5y - \{x - (5z - 2z - y) + 2x - (x - 2y - z)\}]$.

96. Extract the square root of $a^4 - a^3 + \frac{a^2}{4} + 4a - 2 + \frac{4}{a^2}$.

97. A sum of money at simple interest amounted in eight months to \$496 and in fifteen months to \$510; find the sum and the rate of interest.

98. Write out by the binomial theorem the first *five* terms of $\left(\frac{x}{2} - 2y^2\right)^7$.

99. Solve $x - a + \sqrt{x^2 - 2ax} = b$.

100. Solve $\begin{cases} x^2 - y^2 = 8\frac{3}{4}. \\ xy + y^2 = \frac{7}{4}. \end{cases}$

101. Simplify $\sqrt{x^3 - 2x^2y + xy^2}$, $5\sqrt{\frac{3}{5}}$, $\sqrt[6]{8a^3b^6}$, $\sqrt{a^2b - a^3}$, $(16a^4b^5 - 8a^6b^3)^{\frac{1}{3}}$.

102. Divide $x - y$ by $x^{\frac{1}{4}} + y^{\frac{1}{4}}$.

103. The difference of two numbers is 3; the difference of their cubes is 513; find the numbers.

104. Define *literal equation*, *surd*, *binomial*, *similar terms*, *power*.

Commissioners' Certificate.

105. (a) What is the product of m used as a factor c times? (b) What is the per cent. loss on an article bought for fifty cents and sold for n cents? (c) How many tile n inches long and w inches wide will be required to cover a plot o feet long and e feet wide?

106. Multiply $a^2 - 2ab + b^2 + c^2$ by $a^2 + 2ab + b^2 - c^2$.

107. Find the prime factors of

(a) $xy - 2my + 2mn - nx$

(b) $a^2 + 10a - 56$.

(c) $28ab^2 - 28ab + 7a$.

108. (a) Reduce to the form of a whole or mixed number

$$\frac{a^2 + am - 5}{a + m}.$$

(b) Simplify $\frac{1 + a^2}{1 - a^2} + \frac{1 - a^2}{1 + a^2} - \frac{3(1 + a^4)}{1 - a^4}$.

109. (a) What is the reciprocal of a number? Express the reciprocal of $\frac{1}{b}$ in its simplest form. (b) State an axiom which justifies the transposition of a term from one member of an equation to the other.

110. (a) $\frac{9-2z}{3} - \frac{5z+8}{6} + 10 = \frac{3z}{5}$. Solve.

(b) $\left\{ \begin{array}{l} 4x + 12y = 5. \\ x + 2y = 1. \end{array} \right\}$ Solved by substitution.

111. A farmer puts sheep in four pens; in the first he places 4 less than one-third; in the second, 2 more than one-fourth; in the third, 3 more than one-fifth; and in the fourth pen he puts 25 sheep. How many sheep has he?

112. An army corps of 12,850 men was formed into two squares, one of which had 10 men on a side more than the other. How many men were there in each square?

113. Find the square root of $81 - 18m + m^2 + 18n - 2mn + n^2$.

114. Simplify (a) $\sqrt{24a^6b^4c^{3n}}$. (b) $\sqrt{\frac{3}{5}}$. (c) $\sqrt{75} + \sqrt{12}$.
(d) Express without negative exponents $\frac{a}{a^{-3}b^2}$.

115. A man works a weeks at b dollars a week, and his son works c weeks at d dollars a week. With the money they pay for m tons of coal at p dollars a ton, and have s dollars left. Write the equation expressing this condition.

116. Multiply $x^4 + 2x^3y + 4x^2y^2 + 8xy^3 + 16y^4$ by $x - 2y$.

117. Find the prime factors of (a) $x^2 - 10x + 9$; (b) $2ac + bc + 2ad + bd$; (c) $a^2b^2c - ab^2c^2 + a^3bc^3$.

118. Multiply $\frac{x+1}{x-1}$ by $\frac{x+2}{x^2-1}$ by $\frac{x-1}{(x+2)^2}$.

$$119. \begin{cases} \frac{x}{a} + \frac{y}{b} = 2 \\ bx - ay = 0 \end{cases}$$

Solve for the values of x and y .

120. A father gives to his five sons \$1000, which they are to divide so that each elder son shall receive \$20 more than his next younger brother. What is the share of the youngest? (Give algebraic statement and solution.)

121. A. worked 10 days, B. 4 days, C. 3 days, and their wages amounted to \$29; at another time, A. worked 9 days, B. 8 days, and C. 6 days, and their wages amounted to \$36; a third time, A. worked 7 days, B. 6 days, and C., 4 days, and their wages amounted to \$27. How much did each earn in one day? (Give algebraic statement and solution.)

122. The product of two numbers is 126, and the quotient of the greater divided by the less is $3\frac{1}{2}$. What are the numbers? (Give algebraic statement and solution.)

123. Find the square root of $4x^4 + 5x^2 - 2x - 4x^3 + 1$.

124. (a) Reduce to simplest form (*i. e.*, with an integer under the radical) $\sqrt{\frac{5}{7}}$. (b) Express without negative or fractional exponents $5a^{-2}x^{\frac{1}{2}}$. (c) Expand $(-3a^{\frac{1}{2}}b^2)^4$.

125. Find the prime factors of

$$(a) a^6 - 12a^5 + 35a^4.$$

$$(b) 16y^4 - 8y^2 + 1.$$

$$(c) x^3 - a^3.$$

$$(d) x^3 + 2x^2 - x - 2.$$

126. Find two consecutive numbers such that a fifth of the larger shall equal the difference between a third and an eighth of the smaller.

$$127. \begin{cases} \frac{1}{5}(3x - 2y) + \frac{1}{3}(4x - 3y) = x. \\ \frac{4x - 3y}{2} + \frac{2}{3}x - y = 1 + y. \end{cases}$$

Solve for the values of x and y .

128. A. is 4 years older than B., and the sum of the squares of their ages is 976; what are their ages?

129. (a) Multiply $\sqrt{a}$ by $\sqrt[3]{a}$.

(b) Reduce $\sqrt{54am^9}$ to its simplest form.

(c) Divide $(a^2 - b^2)^{\frac{1}{2}}$ by $(a - b)^{\frac{1}{2}}$.

(d) What is the fourth power of $-\frac{3a^2}{x^{\frac{1}{2}}y^{-n}}$?

UNIVERSITY OF PENNSYLVANIA.

Entrance Examination.

130. Factor each of the following expressions:

(a) $p^2 - x^2 + 4x - 4.$

(b) $a^4 - 2ab^3 - b^4 + 2a^3b.$

131. Find the H. C. F. of the following expressions:

(a) $x^2 - 3x, x^2 - 9, x^2 - 6x + 9.$

(b) $x^3 - 5x^2 + 9x - 9, x^4 - 4x^2 + 12x - 9.$

132. Simplify the following expression:

$$\frac{1 - 2x}{1 - x^3} \div \frac{1 - 2x + x^2 - 2x^3}{1 + 2x + 2x^2 + x^3}.$$

133. Solve the equation: $\frac{x + a}{x - b} = \frac{(2x + a)^2}{(2x - b)^2}.$

134. Solve the following system:

$$\begin{cases} \frac{2x + 1}{5} - \frac{3y + 2}{7} = 2y - x. \\ \frac{3x - 1}{4} + \frac{7y + 2}{6} = 2x - y. \end{cases}$$

135. The circumference of the front and hind wheels of a wagon are 2 and 3 yards respectively. What distance has the wagon moved when the front wheel has made 10 revolutions more than the hind wheel?

136. (a) Simplify $\sqrt[3]{(1-x)} \times \sqrt[3]{(x-1)}$.

(b) Express $\frac{3\sqrt[3]{5}-2\sqrt[3]{2}}{2\sqrt[3]{5}-3\sqrt[3]{2}}$ as an equivalent fraction having a rational denominator.

137. (a) Simplify $\left(\frac{2x^{\frac{4}{5}}}{3a^{-2}b^2}\right)^{-5}$ and

(b) $(a^{\frac{1}{3}} + x^{-2})(a^{\frac{1}{3}} - x^{-2})$, and write the results in terms of positive integral powers and roots.

138. Solve the following equations:

(a) $(2x+1)(x+2) = 3x^2 - 4$.

(b) $x^2 - (3a+2b)x + 6ab = 0$.

(c) $\frac{x-1}{\sqrt{x+1}} = 4 + \frac{\sqrt{x-1}}{2}$.

139. (a) If r_1 and r_2 are the roots of the equation $ax^2 + bx + c = 0$, find $r_1 + r_2$ in terms of the coefficients.

(b) Form the equation whose roots are 7, -3.

(c) The equation whose roots are $1 + \sqrt{7}$, $1 - \sqrt{7}$.

140. Solve the simultaneous equations

$$x^2 + y^2 = 74,$$

$$x - y = 2.$$

141. Write the expansion of $\left\{x + \frac{1}{x^2}\right\}^5$ and simplify the result.

142. Find the cube root of $1 - x - \frac{7}{6}x^2 + \frac{26}{27}x^3 + \frac{7}{12}x^4 - \frac{x^5}{4} - \frac{x^6}{8}$.

143. Multiply $2x^{\frac{2}{3}} - 3x^{\frac{1}{3}} - 4 + x^{-\frac{1}{3}}$ by $3x^{\frac{4}{3}} + x - 2x^{\frac{2}{3}}$.

144. Solve $\sqrt{x + \sqrt{x}} + \sqrt{x + \sqrt{x}} = \frac{a\sqrt{x}}{\sqrt{x + \sqrt{x}}}$.

145. Reduce $x^{\frac{1}{3}} - 1 + \frac{x^{\frac{1}{3}}}{x^{\frac{1}{3}} - 1} + \frac{1}{x^{\frac{1}{3}} + \frac{1}{x^{\frac{1}{3}} + 1}}$.

146. Simplify $\frac{\sqrt{xa^{-1}}}{b^{\frac{1}{2}}y^{-\frac{3}{4}}} \div \frac{xa^{-\frac{3}{2}}}{b^2y^{-\frac{1}{2}}}$.

147. A number of two digits is equal to seven times the sum of the digits; show that if the digits be reversed, the number will be equal to four times the sum of the digits.

148. Solve

$$\frac{1}{x} + \frac{1}{y} = 2.$$

$$xy + \frac{1}{x} + \frac{1}{y} = 8.$$

149. Find four numbers in arithmetical progression such that the sum of their squares shall be 120, and that the product of the first and last shall be less than the product of the other two by 8.

150. Simplify the following expression:

$$(a + b + c)^3 - 3(a + b)(b + c)(c + a).$$

151. Factor the following expressions:

$$(a) \ 10(a + b)^2 + 7c(a + b) - 6c^2.$$

$$(b) \ (a + b)^2 + 1 - 2(a + b + 1).$$

152. Find the H. C. F. of the following expressions:

$$(a) \ x^2 - 2a^2 - ax, \ x^2 - 4a^2, \ x^2 - 6a^2 + ax.$$

$$(b) \ x^3 + x^2 - 5x + 3, \ 2x^3 + 7x^2 - 9.$$

153. Add the following fractions :

$$\frac{1}{(a-1)^2 + 3a} - \frac{1}{1-a^3} - \frac{1}{a-1}.$$

154. Simplify the following expression :

$$\frac{1}{1 - \frac{x}{x-1}} - \frac{1}{\frac{x}{x+1} - 1}.$$

155. Solve the following system of equations :

$$\left. \begin{aligned} 3x + 25y - 6z &= 35, \\ 6x - 10y + 21z &= 49, \\ 8x + 15y - 14z &= -4. \end{aligned} \right\}$$

156. Fifteen coins, dollars and quarter-dollars, amount to \$7.50. How many coins of each kind are there?

157. Show that, in the limit, when a becomes zero, one root of the equation $ax^2 + bx + c = 0$ is infinite.

158. Solve the equations :

$$(a) \quad a(x^2 + 1) = x(a^2 + 1).$$

$$(b) \quad \frac{3x-1}{x+2} = \frac{x+3}{x-1}.$$

159. Express without fractional or negative exponents :

$$a^{\frac{2}{3}}, \quad a^{-\frac{3}{4}}, \quad (a^2)^{-\frac{1}{6}}, \quad \frac{1}{(a^{-1\frac{1}{2}})^5}.$$

160. Solve the equations : $\frac{1}{x} - \frac{1}{y} = \frac{1}{2}$, $xy + \frac{1}{x} - \frac{1}{y} = 2\frac{1}{2}$.

161. Rationalize the denominator of the fraction

$$\frac{1}{\sqrt{2} + \sqrt{3} + \sqrt{5}}.$$

162. Expand $(2a - 3a^2)^4$.

163. Find the sum of the following series to five terms :
 $12 + 9 + 6\frac{3}{4} + \text{etc.}$

164. Find the value of $(\sqrt{-2} + \sqrt{-3})(\sqrt{-2} - \sqrt{-3})$.

165. Factor $a^3 - b^3 - a(a^2 - b^2) + b(a - b)^2$.

166. Find lowest common multiple of
 $a^4 - a^3 - 8a + 8$ and $a^4 - 8a^2 + 9a - 2$.

167. Simplify $\frac{m^2 + n^2}{\frac{n}{\frac{1}{n} - \frac{1}{m}}} - m$
 $\frac{n}{\frac{1}{n} - \frac{1}{m}} \times \frac{m^2 - n^2}{m^3 + n^3}$.

168. At what time between 12 and 1 o'clock are the hands of a clock at right angles?

169. Solve the equations
 $x = 4y$ and $\frac{1}{2}(2x + 7y) - 1 = \frac{2}{3}(2x - 6y + 1)$.

170. The sum of two numbers is $25\frac{1}{4}$ and their geometric mean is $\frac{5}{2}$, what are the numbers?

171. Simplify $\sqrt{\left[(x^{\frac{1}{a-b}} \times x^{\frac{1}{a+b}})^{a^4-b^4}\right]^{\frac{1}{a} + \frac{b^2}{a}}}$.

172. Simplify $\sqrt{x^{-\frac{2}{3}}y^{\frac{1}{2}}} \times \sqrt[3]{xy^2} \times \sqrt[11]{y^{12}}$.

173. Prove that $a^0 = 1$.

174. Expand $(2a^{\frac{1}{2}} - b^{\frac{2}{3}}a)^4$ and simplify each term.

175. The third term of an arithmetical progression is 1 and the fifth term is -3 , what is the first? What is sum of five terms?

176. Find the L. C. Dd. of $x^3 - 27$, $x^2 - 15x + 36$,
 $x^3 - 3x^2 - 2x + 6$.

177. Reduce to lowest terms $\frac{(8x^3 - 1)(x^2 - x - 2)}{(6 - 3x)(4x^2 - 1)}$.

$$178. \text{ Simplify } \frac{2a^2c}{\frac{2ac}{a+c} - a} + \frac{2ac^2}{a-c}.$$

$$179. \text{ Simplify } x - 4 + \frac{6}{x+1} \times \frac{1 - \frac{x+5}{x^2-1}}{x - \frac{6}{x-1}}.$$

$$180. \text{ Simplify } \frac{x-y}{x+y} - 1 - \frac{y}{y-x} + \frac{x^2}{x^2-y^2}.$$

$$181. \text{ Solve for } x \quad \frac{3}{x+1} - \frac{x+1}{x-1} = \frac{x^2}{1-x^2}.$$

182. A number of troops being formed into a solid square, it was found that there were 60 over ; but when formed into column with 5 men more in front and 3 less in depth than before, there was just one man wanting to complete it ; find the number of men.

183. Solve for x and y

$$\frac{1}{x} + \frac{2}{y} + \frac{3}{z} = 14.$$

$$\frac{3}{x} + \frac{2}{y} + \frac{1}{z} = 22.$$

$$\frac{5}{x} - \frac{1}{y} - \frac{4}{z} = 18.$$

$$184. \text{ Simplify } \frac{1}{(a-b)-(a-b)x} + \frac{1}{(a+b)+(a+b)x} \text{ and } \frac{x^3-y^3}{x^4-y^4} - \frac{x-y}{x^2-y^2} - \frac{1}{2} \left[\frac{x+y}{x^2+y^2} - \frac{1}{x+y} \right].$$

185. Factor

$$(a) \quad a^3 + a^2b - ab^2 - b^3.$$

$$(b) \quad (p+q)x^3 + (p-q)x^2 - (p+q)x - (p-q).$$

$$(c) \quad (a^2 - 2b^2 - c^2)^2 - 4(b^2 - c^2)^2.$$

186. Solve the simultaneous equations :

$$(a) \frac{9}{x} - \frac{4}{y} = 2 \quad (b) ax + by = 0.$$

$$\frac{18}{x} + \frac{8}{y} = 10 \quad (a-b)x + (a+b)y = 2c.$$

187. Solve for x $\frac{x}{x-a} - \frac{x+2b}{x+a} = \frac{a^2+b^2}{x^2-a^2}$.

188. Simplify $x - [x - 2x - (x - 3x) - (x - 4x)]$. In removing parentheses preceded by the sign $-$, why are the signs of the terms reversed from $+$ to $-$, and from $-$ to $+$?

189. Verify the following identity :

$$(a-b)(a+b-c) + (b-c)(b+c-a) + (c-a)(c+a-b) = 0.$$

Also check by substituting $a = \frac{1}{2}$, $b = -\frac{2}{3}$, $c = \frac{3}{4}$.

190. Simplify the product :

$$(x-a)(x+a)(x^2+a^2)(x^4+a^4)(x^8+a^8).$$

191. Factor the following expressions :

$$2x^2 + 5xy - 12y^2; a^2 - ab - b - 1;$$

$$(x-1)(x-2)(x-3) + (x-1)(x-2) - (x-1).$$

192. Factor the following expressions and write their L. C. Dd. : $a^3 + b^3$, $a^4 + a^2b^2 + b^4$, $a^2 - b^2$.

193. Simplify :

$$(a) \left(\frac{1+x}{1-x} - \frac{1-x}{1+x} \right) \div \left(\frac{1+x}{1-x} - 1 \right).$$

$$(b) \frac{a^2b^2}{c} \div \left(\frac{a^2c^2}{b} \div \frac{b^2c^2}{a} \right).$$

194. Reduce to lowest terms $\frac{(x^2-4)(x^2-3x+2)}{(x^2-4x+4)(x^2+x-2)}$.

195. Simplify

$$\frac{b+c}{(a-b)(a-c)} + \frac{c+a}{(b-c)(b-a)} + \frac{a+b}{(c-a)(c-b)}.$$

196. Solve $\frac{x+1}{2} - \frac{2x-5}{5} = \frac{11x-5}{10} - \frac{x+13}{3}$.

197. Separate a into two parts such that one divided by the other may give b as a quotient and c as a remainder.

198. $\frac{7}{x+3} - \frac{3}{y+4} = 0$, $x(y-2) - y(x-5) = -13$.

Solve for x and y .

199. Find the expansion of $\left(2a^{\frac{1}{2}} - \frac{b}{\sqrt{a}}\right)^6$ and reduce each term to its simplest form.

200. Simplify $\left(\frac{a^{x+y}}{a^y}\right)^x \div \left(\frac{a^y}{a^{y-x}}\right)^{x-y}$.

201. Reduce to a fraction having a rational denominator $\frac{\sqrt{x^2-1} - \sqrt{x^2+1}}{\sqrt{x^2-1} + \sqrt{x^2+1}}$.

202. Simplify $7\sqrt{27} - \sqrt{75} - 24\sqrt{\frac{1}{12}} - 27\sqrt{\frac{1}{27}}$.

203. Solve $\frac{3x+2}{x+3} - \frac{2x+3}{x-3} + \frac{2}{9-x^2} = 0$.

204. Solve $2x^2 - xy = 15$, $x^2 - y^2 = 8$.

205. Simplify

(a) $\left(\frac{3-x}{\sqrt{1-x}} + \sqrt{1-x}\right) \times \frac{1}{\sqrt{2}-\sqrt{x}}$.

(b) $\frac{\sqrt{1+.25^{-4}} + 2 \times 125^{-\frac{4}{3}}}{\frac{9^3 \cdot 18^4}{54^5}}$.

206. Reduce to lowest terms $\frac{x^{3m} - y^{12}}{x^{2m} - y^8}$.

207. Find the expansion of $\left(\sqrt{a} - \frac{1}{\sqrt{a}}\right)^5$ and reduce each term to its simplest form.

208. Simplify

$$2\sqrt[3]{\frac{1}{8}} - 4\sqrt{2} - \sqrt{4\frac{1}{2}} + 3\frac{1}{2}\sqrt{8} - 8\sqrt{\frac{1}{2}} + 6\sqrt{\frac{2}{9}}.$$

209. Reduce to a fraction having a rational denominator

$$\frac{1}{\sqrt{2} - 2\sqrt{3} - \sqrt{5}}.$$

210. Solve $\frac{x - \sqrt{x}}{x + \sqrt{x}} = \frac{4}{x^2 - x}$.

211. Solve for x and y

$$x^2 + y^2 = 5.$$

$$x + y = \frac{3}{2}xy.$$

212. Find the value of m which will make $x^2 - (3m - 1)x + 2m$ exactly divisible by $x - 1$.

213. Solve the equations :

$$\frac{13}{x + 2y + 3} = -\frac{3}{4x - 5y + 6}.$$

$$\frac{3}{6x - 5y + 4} = \frac{19}{3x + 2y + 1}.$$

214. A person has a certain sum, half of which he loans at 5 per cent. interest, and half at $4\frac{1}{2}$ per cent. interest. The half which he loans at 5 per cent. yields him \$60 more interest than the other. What is the amount of his capital?

215. Solve the equations

$$\frac{2x - 1}{x + 1} = \frac{x + 1}{x - 2}; \quad abx^2 - (a^2 + b^2)x + ab = 0.$$

216. Form the equation whose roots are $1 + 2\sqrt{-3}$, $1 - 2\sqrt{-3}$.

217. One root of the equation $x^2 - 4x + 1$ is $2 + \sqrt{3}$; what is the other root?

218. If the side of a square were 53,600 centimetres longer, it would contain 6,553,600 square metres. What is the length of a side?

219. Simplify :

$$\left(\frac{a^{-5}b^{-2}c^2}{a^{-3}b^3c^4}\right)^{-2}; \frac{x^{\frac{1}{2}}y^{-\frac{3}{2}}}{x^{\frac{2}{3}}y^{\frac{1}{4}}}; \frac{a^{2n+1}b^{n-1}}{a^{n+2}b^{1-n}}; \sqrt[3]{\frac{2}{2}}.$$

220. Find the square root of $5 - \sqrt{24}$.

221. Rationalize the denominator of the fraction $\frac{7 + 2\sqrt{6}}{9 - 3\sqrt{6}}$.

222. Find the value of $(\sqrt{3} + \sqrt{-2})(\sqrt{3} - \sqrt{-2})$. Why does $\sqrt{-a}\sqrt{-b} = -\sqrt{ab}$ and not $\sqrt{(-a)(-b)}$?

223. Solve the equations

$$x + y = 20.$$

$$xy = 51.$$

224. Find the sum of

$1 + (1 + b) + (1 + 2b) + (1 + 3b) + \dots + (1 + nb)$
when $b = 2$, $n = 11$.

HAVERFORD COLLEGE.

Examination for Entrance.

225. If $y = \frac{2x - 3}{x^2}$ and $z = \frac{y + 1}{y + 3}$, find y when $x = -1$.

226. Factor

$$x^2 + x - \frac{5}{16} \text{ and } (x - a)(y - b) - (x - b)(y - a).$$

227. Solve the equations

$$(a) \quad 2 \frac{x-2}{x-3} - \frac{1-x}{2+x} = 3.$$

$$(b) \quad \begin{cases} \frac{x}{2} + \frac{y}{3} = 5. \\ 2x - y = 6. \end{cases}$$

$$(c) \quad (x-5)^2 = 4(x-3)^2.$$

228. The difference of two numbers is 5, and the difference of their squares is 15. Find the difference of their cubes.

229. Solve $x^4 - 10x^2 + 1 = 0$.

230. Factor (a) $x^2 - 4x - 60$; (b) $x^6 - y^6$.

231. Simplify
$$\frac{-\frac{(1+x^2)2x - (1-x^2)2x}{(1+x^2)^2}}{\sqrt{\frac{4x^2}{(1+x^2)^2}}}.$$

232. Solve for x and y the simultaneous equations

$$\frac{x+5}{2} + \frac{x+6}{5} = 6,$$

$$\frac{5x-1}{2} - \frac{2y-1}{3} = \frac{14}{3}.$$

233. Solve for x the equation $x + \frac{1}{x} = a + \frac{1}{a}$.

234. A railway train travelled 5 miles an hour slower than usual and was one hour late in making a run of 280 miles. How many miles per hour did it travel?

235. Solve for x and y the simultaneous equations

$$x^2 + xy + y^2 = -21,$$

$$y^2 + 2xy = -15.$$

236. A crew can row 8 miles downstream and back, or 12 miles downstream and half the way back in $1\frac{1}{2}$ hours. What is the rate of rowing in still water and the velocity of the stream?

LAFAYETTE COLLEGE.

Entrance Examination Questions.

237. Define Algebra, Monomial, Similar Terms, Coefficient, Exponent, Degree of an Equation, Simultaneous Equations, Similar Radicals.

238. Give the symbols of quantity, also of relation, in algebra.

239. To what is the zero power of a quantity equal? To what is a quantity with a negative exponent equal? Illustrate each.

240. Divide $x^6 - y^6$ by $x^2 - xy + y^2$.

241. Factor $x^2 - 15x + 56$.

242. Given $\frac{x+1}{2} - \frac{5-x}{4} = 14 - \frac{x+2}{3}$ to find x .

243. Multiply $5x^{\frac{1}{3}}y^{\frac{1}{2}}$ by $-3x^{\frac{2}{3}}y^{\frac{1}{3}}$.

244. Divide $\sqrt[3]{a} \sqrt[3]{b}$ by $\sqrt[3]{a} \sqrt[5]{b}$.

245. Solve $\frac{\sqrt{x} + \sqrt{x-3}}{\sqrt{x} - \sqrt{x-3}} = \frac{3}{x-3}$.

246. Given $x + 2y = 7$; and $x^2 + 3xy + y^2 = 31$; to find x and y .

247. Solve $x^6 - 3x^3 = 40$.

248. Given $\frac{1}{x} + \frac{1}{y} = \frac{8}{15}$; $\frac{1}{y} + \frac{1}{z} = \frac{12}{35}$; $\frac{1}{x} + \frac{1}{z} = \frac{10}{21}$; to find x, y, z .

249. A. and B. can do a piece of work in 12 days, but when A. worked alone he did the same work in 20 days. How long would it take B. to do the same work?

250. When a is added to the greater of two numbers, it is m times the less; but when b is added to the less, it is n times the greater. What are the numbers?

BRYN MAWR COLLEGE.

Examination for Matriculation.

251. Show that $x^3(a-b) - a^3(b+x) + b^3(a+x)$ vanishes when $(x=a+b)$.

252. Simplify $\frac{x^3+27}{x+3} + \frac{x^3-27}{x-3}$.

253. Resolve each of the following into as many factors as you can:

- (i) $(x^4-1)(x^3+1)$;
- (ii) $(a^2+b^2)^2(a^2+c^2) - (a^2+b^2)(a^2+c^2)^2$.

254. Solve the following equations:

- (i) $(4x+1)^2 = 4x^2 + 4x + 9$;
- (ii) $\frac{x+1}{x-4} - 1 = \frac{6}{x-3}$.

255. Is it possible for two numbers to have 56 for their sum and 720 for their product?

256. Prove that the sum of a geometric series of n terms, of which the first term is a and the common ratio is r , is $\frac{a(r^n - 1)}{r - 1}$; hence show that if the first term is a and the n th

term 1, then the sum of the n terms is $\frac{a - br}{1 - r}$.

257. Find the fourth proportional to 3.1, .002, 9.9.

258. Prove that if $a : b :: b : c$, then $a : c = a^2 : b^2 = b^2 : c^2$ and $b = \sqrt{ac}$.

259. State the numerical values of

$$4^{-3}, (.02)^{-1}, \frac{1}{(\sqrt[3]{2})^{-6}} \sqrt[4]{123} \text{ (to 3 places of decimals).}$$

260. Prove that the quadratic equation $x^2 + px + q = 0$ cannot have more than two roots; show that these roots are equal when $p^2 = 4q$. Under what circumstances are they imaginary?

261. Divide $abx^5 - (a^2 + b^2)x^4 + (b^2 + 2ab)x^3 - (b^2 + 2ab)x^2 + (a^2 + b^2)x - ab$ by $bx^2 - ax + b$.

262. Solve the equations

$$(i) \quad \frac{2}{5}(3x - 5) + \frac{1 - 6x}{7} = 3x$$

$$(ii) \quad \left. \begin{aligned} \frac{5x + 18}{30} - \frac{11(y - 25)}{105} &= 11 \\ 11y - 12 &= 21x - 185 \end{aligned} \right\}$$

263. Show that the product of the highest common divisor and the least common multiple of two quantities is equal to the product of the quantities themselves.

264. Find the highest common divisor of $(2a^2 + 3y^2)x + (2x^2 + 3a^2)y$ and $(2x^2 - 3a^2)y + (2a^2 - 3y^2)x$.

265. Reduce to their simplest forms

$$(i) \frac{3x^2 - 5x - 2}{x - 2} + \frac{4 - 9x - 9x^2}{3x - 1} + \frac{x^3 - 8}{x - 2} - x^2;$$

$$(ii) (3^{\frac{1}{3}} + 1)(3^{\frac{2}{3}} - 3^{\frac{1}{3}} + 1);$$

$$(iii) (bc + ca + ab)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - abc$$

$$\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right).$$

266. A man works 365 days and is paid one cent the first day, two cents the second day, three cents the third day, four cents the fourth day, and so on. How many dollars does he receive in all?

267. The sum of two numbers is 11 and the difference of their squares falls short of twenty times the smaller of the two numbers by 47; find the numbers.

268. Form a quadratic equation whose roots are

$$-2 + \sqrt{5} \text{ and } -2 - \sqrt{5}.$$

269. If $a : b :: c : d$, show that

$$3a + 5b : 3c + 5d :: 5a + 3b : 5c + 3d.$$

270. Divide 52 into three parts in geometric progression, these parts being such that the sum of the first and third is $3\frac{1}{3}$ times as great as the second.

271. Find the fourth power of $a^{\frac{1}{4}} + a^{-\frac{1}{4}}$.

272. A horse and carriage were bought for \$450. The horse was sold at a gain of 5 per cent., the carriage at a gain of 20 per cent., making the whole gain 10 per cent. Find the original cost of the horse.

273. Solve the equations

$$\left. \begin{aligned} (i) \quad & \frac{x-3y}{4} - \frac{2x-8y}{7} = 2 \\ & \frac{3x-y}{2} - \frac{8x-2y}{5} + 1 = 0 \end{aligned} \right\}$$

$$(ii) \quad \left. \begin{aligned} x+y &= 2 \\ x^3+y^3 &= 2 \end{aligned} \right\}$$

274. Express as a single rational fraction in its lowest terms $\frac{1}{(a-b)^2} + \frac{1}{(a+b)^2} + \frac{1}{a-b} - \frac{1}{a+b}$.

275. Reduce to its simplest form $\frac{(y-z)^2 + (z-x)^2 + (x-y)^2}{(y-z)(z-x)(x-y)} + 2 \left\{ \frac{1}{y-z} + \frac{1}{z-x} + \frac{1}{x-y} \right\}$.

276. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, show that each is equal to $\frac{la+mc+ne}{lb+md+nf}$.

277. If $a:b=c:d$, prove that

$$\frac{a(a+d)}{c(b+c)} = \frac{a^{n+1}d^{n-1} + b^nc^n}{a^{n-1}c^2d^{n-1} + a^nd^n}.$$

278. Sum the following series :

(i) $201 + 199 + 197 + 195 + \dots$ to 101 terms ;

(ii) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$ to infinity.

279. (i) Prove that when m and n are positive integers,
 $a^m \times a^n = a^{m+n}$.

(ii) How are meanings assigned to $a^{\frac{1}{2}}$, $(a^{\frac{1}{2}})^{\frac{1}{3}}$, a^0 ,
 $(a^{-2})^{-3}$, $a^{-2}a^{-3}$?

280. Simplify $\left\{ \frac{\sqrt{a+b}}{\sqrt{a-b}} - \frac{\sqrt{a-b}}{\sqrt{a+b}} \right\} \frac{\sqrt{a^3+b^3}}{\sqrt{a^2-ab+b^2}}$.

281. Find a quantity such that when it is subtracted from

each of the quantities a, b, c , the remainders are in continued proportion.

282. Two spheres of lead of radii 3 and 4 feet are united into a single sphere. Find the radius of the new sphere, given that the volume of a sphere varies as the cube of its radius.

PENNSYLVANIA STATE COLLEGE.

Entrance Examination.

283. Factor $x^2 - 9xy + 20y^2$, $a^2 + x^4 - 1 - 2ax^2$
 $a^3 + a + b^3 + b$, $(a + 3b)^3 - (3a + b)^3$.

284. Solve $x^3 + y^3 = 72$,
 $x^2 - xy + y^2 = 12$.

285. Solve in two ways, $2x^2 + \frac{1}{3}x + 5 = x$; and show by formula how to determine the character of the roots of the following equations without solving them: $x^2 - x + 1 = 0$, $ax^2 + bx + c = 0$.

286. Make an equation whose roots shall be 1,
 $-\frac{1}{2} + \frac{1}{2}\sqrt{-3}$, $-\frac{1}{2} - \frac{1}{2}\sqrt{-3}$.

287. Rationalize and then compute to two places of decimals $\frac{1}{\sqrt{2} - \sqrt{3}}$.

288. Solve the equation $ax + by = c$,
 $a'x + b'y = c'$.

289. Expand by the binomial theorem $(2\sqrt{x} - x^{-1})^4$.

290. Solve $\sqrt{3x + 7} - 2 - \sqrt{x + 1} = 0$.

PRINCETON.

School of Science.

291. Find the factors of

$$x^2y + 3xy^2 - 3x^3 - y^3; y^2 - c^2 + 2cx - x^2; 81a^4 - 16b^4.$$

292. Simplify the following expressions :

$$\left(\frac{a^{-\frac{1}{2}}}{4c^2}\right)^{-\frac{1}{2}}; (\sqrt[b]{x^b} \div \sqrt[a]{x})^{\frac{1}{1-a}}; \frac{\sqrt{7} + \sqrt{2}}{9 + 2\sqrt{14}}.$$

293. Extract the square root of

$$4x^n + 9x^{-n} + 28 - 24x^{-\frac{n}{2}} - 16x^{\frac{n}{2}}.$$

294. Solve the following equations :

$$\begin{cases} px - qy = r \\ rx - py = q \end{cases} \quad \begin{cases} \sqrt{x+y} + \sqrt{x-y} = 4 \\ x^2 - y^2 = 9. \end{cases}$$

295. Show that the roots of the quadratic equation $x^2 + px + q = 0$ are both imaginary when $p^2 - 4q$ is negative.

296. If $a : b = c : d$, show that $a^2 + ab : c^2 + cd = b^2 - 2ab : d^2 - 2cd$. If $a : b = b : c = c : d$, show that $b + c$ is a mean proportional between $a + b$ and $c + d$.

297. The area of a circle varies as the square of its radius. Show that a circle of 5 feet radius is equal to the sum of a circle of 3 feet radius and another of 4 feet radius.

298. If a, b, c , and d are in arithmetic progression, show that $a + d = b + c$. The sum of the first and second terms of a geometric progression is 72, and the sum of the third and fourth is 8. What is the first term?

299. Find the factors of

$$250(a - b)^3 + 2, 1 - a^2x^2 - b^2y^2 + 2abxy, x^4 + y^4 - 18x^2y^2.$$

$$300. \text{ Simplify } (a^{n^2-1})^{\frac{n}{n+1}} + \frac{\sqrt[n]{a^{2n}}}{a}, \frac{3 + \sqrt{5}}{\sqrt{5} - 2}$$

$$\frac{2^n \times (2^{n-1})^n}{2^{n+1} \times 2^{n-1}} + \frac{1}{4^{-n}}$$

$$301. \text{ Extract the square root of } 25x^{\frac{4}{3}} - 30x + 16 - 24x^{\frac{1}{3}} + 49x^{\frac{2}{3}}.$$

302. Solve the equations :

$$\begin{cases} ax = by \\ bx + ay = c \end{cases}$$

$$\begin{cases} y + \sqrt{x^2 - 1} = 2 \\ \sqrt{x + 1} - \sqrt{x - 1} = \sqrt{y} \end{cases}$$

$$303. \text{ Solve the equation } \frac{x - a}{x + a} + \frac{x + a}{x - a} = 5.$$

$$304. \text{ Solve in positive integers } 6x - 13y = 1.$$

305. If the ratios $a : b$, $c : d$, and $e : f$ are equal, show that each is equal to the ratio $a + c + e : b + d + f$.

306. If $a : b = c : d$, show that $ab + cd : ab - cd = a^2 + c^2 : a^2 - c^2$.

$$307. \text{ Insert 16 arithmetic means between } -7\frac{1}{5} \text{ and } -6\frac{2}{5}.$$

$$308. \text{ Sum to infinity the progression } \frac{1}{2}, \frac{1}{4}, \frac{1}{8} \dots \dots \dots$$

309. If a, b, c, d are four quantities in geometric progression, show that $b + c$ is the geometric mean between $a + b$, and $c + d$.

$$310. \text{ Expand to five terms by the binomial theorem } (x - x^{-1})^8.$$

$$311. \text{ Factor } 28x^4y + 64x^3y - 60x^2y \cdot 1 - x^2 - 2xy - y^2; a^3 + b^3 + a + b.$$

312. Simplify $\left\{ \frac{a^{p-q}}{\sqrt[q]{a^{q^2-pq}}} \times a^{2(p-q)} \right\}^n$.

313. Extract the square root of $9x - 12x^{\frac{1}{2}} + 10 - 4x^{-\frac{1}{2}} + x^{-1}$.

314. Solve the equations :

$$(a) \quad \begin{cases} qx - rb = p(a - y) \\ q \frac{x}{a} + r = p \left(1 + \frac{y}{b} \right) \end{cases}$$

$$(b) \quad \begin{cases} \frac{1}{x^2} + \frac{1}{y^2} = \frac{45}{4} \\ \frac{1}{x} - \frac{1}{y} = \frac{3}{2} \end{cases}$$

315. If the equation $x^2 + 2(1+k)x + k^2 = 0$ has equal roots, what is the value of k ?

316. Find the general solution in positive integers of $8x - 21y = 33$.

317. If a, b, c, d are in continued proportion, prove that $b + c$ is a mean proportional between $a + b$ and $c + d$.

318. Insert four geometric means between 160 and 5.

319. How many terms must be taken of the arithmetic progression 42, 39, 36 . . . to make 315?

320. Expand to four terms by the binomial theorem $(2ax^2 - x^3)^5$.

321. Divide $(p^{\frac{1}{2}} - q)$ by $(p^{\frac{1}{10}} - q^{\frac{1}{5}})$.

322. Find the highest common factor of

$$2x^4 - 7x^3 + 11x^2 - 8x + 2 \text{ and}$$

$$2x^4 + x^3 - 9x^2 + 8x - 2.$$

323. Simplify $[(x^2)^{\frac{1}{7}} (x^3)^{-\frac{1}{2}}]^{-14} [(y^{\frac{1}{3}})^{-2} (y^{\frac{1}{2}})^7]^{-6} \div (x^3 y^{-3})^7$.

324. Solve for x and y , $5(x - 2y) - (x - y) = -24$.

$$11(2x + 3y) + (2x - y) = 200.$$

325. Extract to three decimal places the square root of 394.1.

326. Solve (a) $lx^2 + mx + n = 0$.

(b) $(3x - 2)^2 + (x - \frac{2}{3}) = 84$.

327. Solve for x and y , $xy - 5x = 1$.

$$7x - y = 1.$$

328. By use of the formulæ for arithmetic progressions find the sum of all the numbers between 1 and 207 which are divisible by 5.

329. By use of the binomial theorem expand $(1 - \frac{2}{3}x)^{\frac{3}{2}}$.

330. Resolve into prime factors $(7x + 4y)^2 - (2x + y)^2$.

331. Reduce to its simplest form $(1 - \sqrt{a} - b\sqrt{a} + ab) \div (b\sqrt{a} - b)$.

332. What must be the value of n in order that $(2a + n) \div (3n + 69a)$ may be equal to $\frac{1}{3\frac{1}{3}}$ when a is $\frac{1}{3}$?

333. Simplify $\left\{ \left(\frac{x + (x^2 - a)^{\frac{1}{2}}}{2} \right)^{\frac{1}{2}} + \left(\frac{x - (x^2 - a)^{\frac{1}{2}}}{2} \right)^{\frac{1}{2}} \right\}^2$.

334. Find to four decimal places the square root of 932.131.

335. Find sum to infinity of $-3, +\frac{1}{2}, -\frac{1}{12}, \dots$

336. Solve $\frac{5}{x+2} + \frac{2}{x} = \frac{14}{x+4}$.

337. Solve $3x^2 - 2y^2 = 115$.

$$2x - 3y = 2.$$

338. Find the least common dividend of $(x^3 - y^3)(x - y)^3$, $(x^2 - y^2)^2(x - y)^2$, $(x + y)(x^3 + y^3)$.

339. Find the value of

$$\frac{a^2 - bc}{(a + b)(a + c)} + \frac{b^2 - ca}{(b + c)(b + a)} + \frac{c^2 - ab}{(c + a)(c + b)}.$$

340. Simplify $\frac{\{(a^3)^{\frac{1}{3}}(a^5)^{\frac{1}{5}}\}^{91}}{\{(b^{\frac{1}{5}})^7(b^{\frac{1}{3}})^{13}\}^{15}} \div \left\{\left(\frac{a}{b}\right)^5\right\}^{13}$.

341. The sum of two numbers, x and y , is 240 and their difference is equal to $\frac{1}{3}$ of the greater; find the numbers.

342. Find to three decimal places the square root of 253.481.

343. Solve for x , $2x + 14 + \frac{7}{x} = 0$.

344. Solve for x and y (finding the four sets of values of x and y), $x^2 + y^2 = 170$.

$$xy = 13.$$

345. Find the sum of 10 terms of the progression $\frac{2}{3}, \frac{1}{2}, \frac{3}{8}, \dots$

AMHERST COLLEGE.

Entrance Examination.

346. Factor (1) $3x^2 - 17x + 10$; (2) $8x^3 + 27$.

347. How many values of y make $y + \frac{1}{y} = 4$? Why? Find these values and verify the result.

348. Solve $\left\{ \begin{array}{l} \frac{x-2}{3} - \frac{y+5}{2} = 0. \\ \frac{2x-7}{3} - \frac{13-y}{16} = 0 \end{array} \right\}$.

349. Divide $a^{\frac{4}{3}} - 2 + a^{-\frac{4}{3}}$ by $a^{\frac{2}{3}} - a^{-\frac{2}{3}}$.

350. A man invests two-thirds of his capital at 4 per cent., one-fourth at 3 per cent., and the remainder at 2 per cent. His income from the investment is \$215. Find his capital.

351. Find the sum of 6 terms of the series $2 - 3 + \frac{9}{2}$

352. The roots of an equation are $\frac{-c + \sqrt{c^2 - ab}}{a}$ and $\frac{-c - \sqrt{c^2 - ab}}{a}$. Find the equation.

353. (1) Show that

$$\frac{1}{(x-y)(x-z)} - \frac{1}{(x-y)(y-z)} + \frac{1}{(z-x)(z-y)} = 0$$

is an identity.

(2) Put $x=3$, $y=2$, $z=0$ in the preceding expression and find its value.

(3) Does the fact that the value thus obtained is 0 prove (1) an identity? Give the reason for your answer.

354. Simplify $2a - [a - \{b - (3b - 2a - b)\}] - (b - a)$.

355. Find the L. C. Dd. of $x^2 + x - 6$, $x^2 - 4x + 4$, and $x^3 - 9x$.

356. Simplify $[(x^{a-b})^{\frac{a-b^2}{a}}]^{\frac{a}{a+b}}$.

357. Find two numbers in the ratio 5 : 3, the difference of whose squares is 400.

358. Extract the square root of $17 + 4\sqrt{15}$.

359. (a) What is the *root* of an equation?

(b) Is $\frac{2}{3}$ a root of the equation $3x^2 + 5x + 2 = 0$?

Is $-\frac{2}{3}$? Is -1 ?

360. Find the sum of the following series to infinity:
 $\frac{3}{4} + \frac{1}{2} + \frac{1}{3} + ...$

361. Solve $\begin{cases} x^2 + xy = 12, \\ xy + y^2 = 4. \end{cases}$

362. Find the fourth term of $\left(x + \frac{1}{2x}\right)^8$, and express the result in its simplest form.

363. A hare pursued by a hound takes 4 leaps while the hound takes 3 ; but 2 of the hound's leaps are equal to 3 of the hare's. If the hare has a start equal to 60 of her own leaps, how many leaps must the hound take to catch the hare?

YALE COLLEGE.

Entrance Examination.

364. Find the value of $\frac{4\sqrt{12.5} - 2\sqrt{.5}}{6\sqrt{18}}$.

365. Factor $a^3 - 8b^{15}$, $81x^4 - y^8z^{12}$.

366. Rationalize the denominator of $\frac{\sqrt{1 - \frac{1}{2}\sqrt{2}}}{\sqrt{1 - \frac{\sqrt{5}}{3}}}$.

367. Solve $\frac{6}{x} - \frac{7}{y} = 2$.
 $\frac{2}{x} + \frac{14}{y} = 3$.

368. Simplify $\frac{b}{\sqrt{a}} \cdot \sqrt[3]{ac} \cdot \frac{\sqrt[4]{c^3}}{\sqrt{b}} \cdot \frac{\sqrt[3]{b^{-1}}}{a^{-\frac{1}{6}}}$.

369. A bill of \$12.50 is paid with quarter and half dollars. Twice the number of half dollars exceeds three times the number of quarters by 10. Find the number of each.

370. Solve $x^2y^2 + 28xy + 192 = 0$.

$$x - y = 8.$$

371. Find a geometric progression in which the sum of the first two terms is $2\frac{2}{3}$ and the sum to infinity is $4\frac{1}{6}$.

372. The price of photographs is raised \$3.00 per dozen, and customers consequently receive 10 less than before for \$5.00. Find the new and old prices.

373. Factor (a) $3x^3 - 12x^2y^2 - 4y^2 + 1$.

(b) $3a + 3b - 6\sqrt{ab}$.

(c) $8 + 2\sqrt{15}$.

374. Rationalize the denominator of

$$\frac{1}{2\sqrt{3} + \sqrt{-2} - \sqrt{6}}.$$

375. Simplify $2\sqrt{\frac{8}{3}} + 2\sqrt{\frac{3}{2}} + \sqrt{\frac{27}{8}}$.

376. Solve $\frac{1}{\sqrt{a-x} + \sqrt{a}} + \frac{1}{\sqrt{a-x} - \sqrt{a}} = \frac{\sqrt{a}}{x}$.

377. A man starts on a bicycle ride at the rate of 10 miles an hour, intending to ride 20 miles. Owing to an accident he walks part of the way at the rate of 4 miles an hour, and finds himself one hour and a half late. How far does he walk?

378. Find values of x and y which satisfy the equations

$$\frac{x - 3y}{2} + \frac{y - 3x}{2} = 8.$$

$$\frac{\frac{1}{5}x + \frac{3}{4}y}{\frac{1}{2}x - \frac{11}{3}y} = -\frac{7}{30}.$$

379. Solve $\sqrt{x+1} : \sqrt{2x} = \frac{1}{x} : \frac{2}{x+4}$.

380. Solve $\left(x^{-2} + \frac{1}{4}\right)^{-3} = 27$.

381. What distance is passed over by a ball which falls from a height of 60 feet, and at every fall rebounds $\frac{1}{3}$ the distance from which it fell?

382. How many different sounds can be made by striking 10 keys of a piano 3 at a time?

383. There are two fractions: the numerator of the first is the square of the denominator of the second, and the numerator of the second is the square of the denominator of the first. The sum of the fractions is $\frac{35}{6}$ and the sum of the denominators is 5. Find the fractions.

MOUNT HOLYOKE COLLEGE.

Entrance Examination.

384. Factor $(a+b)^4 - 81c^4$.

$$a^3 + a^2b - 6ab^2.$$

$$a^2b^2 - a^2 - b^2 + 1.$$

$$(a^3 + 3a)^2 - (3a^2 + 1)^2.$$

385. Solve for x, y , and z , $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = a$,

$$\frac{b}{y} - \frac{c}{z} + \frac{a}{x} = b,$$

$$\frac{a}{x} - \frac{b}{y} - \frac{c}{z} = c.$$

386. Simplify

$$2\sqrt{\frac{3}{2}} + 3\sqrt{\frac{2}{3}} - \left(\frac{2}{3}\sqrt[3]{\frac{1}{5}} \div \sqrt[3]{\frac{25}{4}}\right) \left(\frac{1}{2}\sqrt{6} - \sqrt{24}\right).$$

387. Solve the equations :

$$(a) \begin{cases} 27x^3 + y^3 = 152, \\ 3x^2y + xy^2 = 40. \end{cases}$$

$$(b) x^{\frac{1}{a}} - \frac{5}{x^{\frac{1}{a}}} = 4.$$

388. Find a fourth proportional to a^3 , ab , $5a^2b$. The arithmetic mean of two numbers is $20\frac{1}{2}$, the geometric mean is 20. Find the numbers.

389. Solve the equation $ax^2 - cx + \frac{b}{4} = 0$; state the condition (a) for equal roots, (b) for imaginary roots. Form the equation whose roots are $\frac{1}{2} \pm \sqrt{2}$.

390. (a) Extract the square root of $75 - 12\sqrt{21}$.

$$(b) \text{ Expand } \left(\frac{a^{\frac{1}{2}}}{2} - \frac{2b^2}{3a^{\frac{1}{3}}}\right)^4.$$

391. Simplify :

$$(a) \frac{2}{x^2 + xy + y^2} - \frac{4x(x^3 + y^3)}{x^6 - y^6} + \frac{y^{-2}(x - y)^{-2}}{x^{-2}} - \frac{x^2}{x^3y - y^4}.$$

$$(b) \left(x^{1+\frac{a}{b}}\right)^{-\frac{a}{a+b}} \div \sqrt[3]{\frac{x^{2a}}{(x^{-1})^{-a}}}.$$

392. If $x : a :: y : b$, prove

$$mx + y : ma + b :: px + y : pa + b.$$

393. Rationalize the denominator of $\frac{\sqrt{-3} + 3}{\sqrt{-4} - 2\sqrt{3}}$.

394. Solve the following equations :

$$(a) \quad 2x^2 - 3\sqrt{x^2 + 2x + 14} + 4x - 49 = 0.$$

$$(b) \quad \begin{cases} x^{\frac{1}{2}} + y^{\frac{1}{2}} = 5, \\ 6(x^{-\frac{1}{2}} + y^{-\frac{1}{2}}) = 5. \end{cases}$$

395. Sum the series $6 - 4 + \frac{8}{3} - \text{etc.}$ to 7 terms, and find the limit when continued to infinity.

396. (a) Expand $(2\sqrt{-4} - 2\sqrt{-5})^3$.

(b) Find the highest common factor of $x^6 - y^6$, $2x^4 + x^2y^2 - 3y^4$, $2by^2 + 3ax^2 - 3ay^2 - 2bx^2$.

397. Simplify :

$$(a) \quad 2\sqrt[3]{3} (\sqrt[3]{9} - 2\sqrt[3]{2\frac{2}{3}} + 4\sqrt[3]{\frac{1}{3}} - 3\sqrt[3]{2}).$$

$$(b) \quad \frac{b^2}{a^8c^2} \times \frac{a^{-1}b^{-2}}{ab^{-3}} \times \frac{ab^{2-1}c^{-2}}{b^4} \div \left(\frac{a^{-2}b^{-1}}{a^2} \right)^2.$$

398. Insert between 6 and 16 two numbers such that the first three numbers may be in arithmetic progression, and the last three in geometric progression.

399. Solve the equations :

$$(a) \quad \sqrt{x} + \sqrt{x+4} = \frac{2}{\sqrt{x}}.$$

$$(b) \quad \begin{cases} \frac{1}{x^2} - \frac{1}{4y^2} = 3, \\ \frac{1}{x^2} - \frac{1}{xy} + \frac{1}{4y^2} = 9. \end{cases}$$

400. Solve the equation $x^2 + 4bx + c = 0$; state the condition (a) for equal roots, (b) for real roots.

THE JOHNS HOPKINS UNIVERSITY.

Matriculation Examination.

401. Find least common dividend of

$$a^4 - 1, a^3 + a^2 + a + 1, a^3 - a^2 + a - 1, a^2 + 1.$$

402. Simplify

$$\frac{x+2}{x^3-x} - \frac{1}{2x+2} - \frac{3}{2x-2} + \frac{2}{x}; \text{ and } \frac{1}{x + \frac{1}{1 + \frac{x+1}{3-x}}}.$$

403. Factor $ax^3 - 2ax^2 - 3x + 6$; also $a^2 - b^2 + 8a + 16$.

Is $x^{17} + y^{17}$ divisible by $x + y$? by $x - y$? Give reasons for your answers.

404. Divide $1 + 2x^{\frac{2}{3}} - 7x^{\frac{4}{3}} - 16x^2$ by $1 + 2x^{\frac{1}{3}} + 3x^{\frac{2}{3}} + 4x$.

405. An Artesian well supplies a manufactory. The water is drawn out each week-day from 3 A. M. until 6 P. M. twice as rapidly as it runs into the well. If the well contained 2250 gallons of water on Monday morning at 3 A. M. and was emptied precisely at 6 P. M. a week from the following Thursday, how many gallons flowed into the well per hour?

406. Solve $2x^2 + 3x + 4 = 0$; also $\frac{x}{x-1} = \frac{3}{2} + \frac{x-1}{2}$.

407. Express $\sqrt{27 - 10\sqrt{2}}$ in terms of simple surds.

Simplify $\frac{1}{\sqrt{-1}}$ and $\frac{8}{\sqrt{3} + \sqrt{2} + 1}$ so that no surd appears in the denominators.

408. Solve $\begin{cases} x^2 + y^2 + x + y = 18 \\ xy = 6. \end{cases}$

409. Solve

$$x^{\frac{1}{4}} - x^{-\frac{1}{4}} = \frac{3}{2}; \text{ and } \sqrt[4]{a} + \sqrt[4]{x} + \sqrt[4]{a - \sqrt[4]{x}} = \sqrt[4]{x}.$$

410. Each term in a certain infinite decreasing geometric progression is equal to the sum of all that follow it. Find the common ratio.

411. If $a : b :: c : d$, prove that $a^2 + b^2 : c^2 + d^2 :: (a + b)^2 : (c + d)^2$.

VASSAR COLLEGE.

Entrance Examination.

412. Solve $xy = 4 - y^2$
 $2x^2 - y^2 = 17.$

413. Show that

$$(a + b\sqrt{-1})(a - b\sqrt{-1}) = (a + b + \sqrt{2ab})(a + b - \sqrt{2ab}).$$

414. Solve $x - \frac{14x - 9}{8x - 3} = \frac{x^2 - 3}{x + 1}.$

415. Prove that if $a : b = c : d$, $\frac{a + b}{c + d} = \frac{a - b}{c - d} = \frac{a}{c} = \frac{b}{d}.$

416. Expand $(1 + x^2)^7$ by the binominal formula.

417. Find the $(2n)th$ term of the series 1, 3, 5, 7

418 Form the quadratic equation whose roots are $\frac{3}{2}$ and $\frac{2}{3}.$

419. Solve $\sqrt{2x + 6} + \sqrt{2x - 5} = 11.$

420 Produce the formula for *last term* and *sum* in arithmetic and geometric progression.

421. Rationalize the denominator of

$$\frac{\sqrt{x^2 + x + 1} + \sqrt{x^2 + x - 1}}{\sqrt{x^2 + x + 1} - \sqrt{x^2 + x - 1}}.$$

422. Solve $\sqrt{x+4} - \sqrt{x} = \sqrt{x + \frac{3}{2}}$.

OHIO.

Examination for Teacher's Certificate.

423. Solve the equations :

$$(a) (x-5)^3 - 3(x-5)^{\frac{3}{2}} = 40;$$

$$(b) 8(5x-3)^{-\frac{4}{3}} - 6(5x-3)^{-\frac{2}{3}} = -1.$$

424. Expand by binomial theorem $\left(\sqrt[5]{x^2} - \frac{1}{\sqrt{y}}\right)^7$.

425. Solve for x : $x^2 - xy = 27y$.

$$xy - y^2 = 3x.$$

426. If the product of two numbers be added to their difference, the result is 26; and the sum of their squares exceeds their difference by 50. Find the numbers.

427. A woman sells half an egg more than half her eggs. She then sells half an egg more than half her eggs remaining. A third time she does the same, and now she has sold all the eggs. How many had she at first?

428. Factor $a^6 - a^4 - a^2 + 1$; $(x^2 - y^2 - z^2)^2 - 4y^2z^2$;
 $(x^2 - 5x)^2 - 2(x^2 - 5x) - 24$.

429. If $a : b :: (a + c)^2 : (b + c)^2$, prove that c is a mean proportional between a and b .

430. Simplify :

$$(a) \frac{b+c}{(a-b)(a-c)} + \frac{a+c}{(b-c)(b-a)} + \frac{a+b}{(c-a)(c-b)}.$$

$$(b) \frac{1}{1-x} + \frac{1}{1+x} + \frac{2}{1+x^2} + \frac{4}{1+x^4}.$$

431. A man travelled 105 miles. If he had gone 4 miles more an hour, he would have performed the journey in $9\frac{1}{3}$ hours less time. How many miles an hour did he go?

432. Factor $4a^4b - 12a^4b^2 - 16a^4b^3 + 8a^4b^4$; $6x^3 + 4x^2 - 9x - 6$; $c^2d + c^2e + f^2d + f^2e - 2efe - 2efd$; $x^4 + x^2 + 1$; $a^2 - 4a - 45$.

433. A jeweler sold three rings. The price of the first with $\frac{1}{2}$ that of the second and third was \$25; the price of the second with $\frac{1}{3}$ that of the first and third was \$26; and the price of the third with $\frac{1}{2}$ that of the first and second was \$29. What was the price of each?

$$434. \text{ Simplify } \sqrt{\frac{1}{2}} - 2\sqrt{\frac{1}{8}} - \sqrt{\frac{4}{18}}; 3\sqrt{\frac{a+x}{a-x}} \times 4\sqrt[3]{a-x}.$$

435. Expand by binomial theorem $(x^2 - 2x - 1)^5$.

436. Express with positive exponents (a) $\frac{x^{-3}y^{-5}}{a^2b^{-2}}$;

Express with fractional exponents (b) $\sqrt[3]{16a^2x^3}$.

437. Solve for x : $2x^2 - 94x + 420 = 0$; $2(x-2)(x-3) = (x-4)(x-3) + 10$.

438. A certain number of sheep were bought for \$468; but after 8 of them had been reserved, the rest were sold at an advance of \$1 a head, and \$12 were gained on the lot. How many sheep were bought?

439. A., B., C., and D. found a sum of money. They agreed that A. should receive \$4 less than $\frac{1}{3}$, B. \$2 more than $\frac{1}{4}$, C. \$3 more than $\frac{1}{5}$, and D. the remainder, \$25. How much did A., B., and C. receive?

440. Simplify :

$$(a) \sqrt{-a^2} - \sqrt{-4a^2} - \sqrt{-9a^2};$$

$$(b) 2\sqrt{12x^2 + 60xy + 75y^2} - \sqrt{48x^2 - 72xy + 27y^2}.$$

441. Factor (a) $x^2z^2 + ax^2 - a^2z^2 - a^3$;

$$(b) 3a^2 - 6ab + 3b^2 + 6ac - 6bc;$$

$$(c) 49b^4 - 11b^2c^2 + 25c^4.$$

442. Simplify

$$\frac{1}{a(a-b)(a-c)} + \frac{1}{b(b-a)(b-c)} + \frac{1}{c(c-a)(c-b)}.$$

443. A. and C. can do a piece of work in 6 days; B. and C. can do it in 8 days. In what time can they all do it working together, if A. can do $\frac{3}{2}$ as much as B.?

444. Simplify $2\sqrt[3]{3x} + 5\sqrt[6]{9x^2} - \sqrt[3]{27x^3}.$

445. $\frac{3x-1}{\sqrt{3x+1}} - \frac{\sqrt{3x}-1}{2} = 4$, find value of x .

446. $\left(\frac{6}{x} + x\right)^2 + \left(\frac{6}{x} + x\right) = 30$, find value of x .

447. $\left. \begin{array}{l} x^2 + y^2 = 52 \\ x + y + xy = 34 \end{array} \right\}$ find x and y .

448. A boy rode 10 miles on his bicycle, when it broke down, and he was compelled to return on foot. He found that it took 1 hour and 15 minutes longer to walk back than it did to ride out. How fast did he ride, if he walked 4 miles less per hour than he rode?

VIRGINIA.

Examination for State Certificate.

449. (a) Prove that any quantity having a cipher for an exponent is equal to unity.

(b) Factor $y^6 - 2y^3z^3 + z^6$.

450. (a) Find the greatest common divisor of $16a^2 - c^2$, and $16a^2 - 8ac + c^2$.

(b) Find the least common dividend of $(a^2 - x^2)$, $4(a - x)$, and $(a + x)$.

451. Simplify the fraction
$$\frac{5c + \frac{a-b}{2x}}{5c - \frac{a-b}{2x}}$$

452. Given $\frac{3x}{5} + 2\frac{1}{2} + 11 = \frac{x}{4} + 17$, to find x .

453. A. can do a piece of work in 12 days, and B. can do the same in 24 days: how many days will be required if they work together?

454. Given
$$\left\{ \begin{array}{l} x + y + z = 26 \\ x - y = 4 \\ x - z = 6 \end{array} \right\}$$
 to find x , y , and z .

455. (a) Multiply $\sqrt{a} + \sqrt{c}$ by $\sqrt{a} + \sqrt{c}$.

(b) Divide $\sqrt{\frac{x}{y}}$ by $\sqrt{\frac{a}{b}}$.

456. Given $x + 2 = \sqrt{4 + x} \sqrt{64 + x^2}$ to find x .

457. Given $3x^2 + 5x = 42$, to find the values of x .

458. Find two numbers, such that the less may be to the greater as the greater is to 12, and that the sum of their squares may be 45.

WHEELING, WEST VIRGINIA.

High School Examination.

459. Define the following : Involution, Evolution, Monomial, Binomial, Root, Radical.

460. Define the following : Quadratic Equation, Incomplete Quadratic, Complete Quadratic.

461. Give three common cases where Simultaneous Quadratic Equations may be solved as equations of the second degree.

462. Define the following : Ratio, Proportion, Variation, Progression, Arithmetic Progression, Geometric Progression.

463. Find the square root of $1 - 10x + 27x^2 - 10x^3 + x^4$.

464. Find the cube root of $x^3 - 6x + \frac{12}{x} - \frac{8}{x^3}$.

465. Simplify $2\sqrt{-1} + 2\sqrt{-9} - 3\sqrt{-4}$.

466. Solve the following : $x^2 - \frac{x}{2} - 5 = 0$.

467. The number of dollars A. has is to the number B. has as 9 to 5. By obtaining one-half of As.' money, B. will have \$2 more than A. had at first. How much money has each ?

468. A man began saving by putting by 1 cent on New Year's Day, 2 cents the next day, 3 cents the next day, and so on. In how many days would he have put by \$98.70 ?

469. Give the law for coefficients and signs in multiplication, and give your method of explaining the law of signs, with illustrations.

470. Give principles used in finding greatest common divisor when the quantities are not easily factored, and find the greatest common divisor of $6x^3 + x^2 - 5x - 2$ and $6x^3 + 5x^2 - 3x - 2$.

471. Explain the force of zero used as an exponent, and determine the value of a^0 .

472. Find the value of x in the equation :

$$\frac{a+x}{b} - \frac{c-y}{d} = \frac{a}{b}$$

and give reasons for each transformation.

473. Give two methods of elimination in equations of two or more unknown quantities, illustrate each with example, and explain steps.

474. Divide $(ax^2)^{\frac{1}{3}}$ by $(xy)^{\frac{1}{7}}$, and explain the operation.

475. Multiply $\frac{x^3 - xy}{3a + 3b}$ by $\frac{a + b}{x - y}$.

476. Solve the equation $x - \frac{26 - x}{2}$ equals $\frac{x - 11}{3}$.

477. A courier starts from a certain place and travels at the rate of $10\frac{3}{5}$ miles an hour. Two hours later a second courier, travelling $13\frac{2}{3}$ miles an hour, is sent to recall the first. In how many hours will the second courier overtake the first?

478. Find the fourth root of $x^4 - 28x^3 + 294x^2 - 1372x + 2401$.

Public School Examination.

479. Define the following : Factor, Coefficient, Exponent, Reciprocal, Equation, Factoring, Monomial, Polynomial, Multiple, Elimination.

480. Find the H. C. F. of $24x^2 - 2ax - 35a^2$ and $48x^3 - 60ax^2 - 84a^2x + 105a^3$.

481. Expand $(3a^2 yn + 2c^3 xn)^2$.

482. Factor $9c^2 - (2a - 3b)^2$.

483. Factor $9x^2 + 6xy - 15y^2$.

484. Reduce to fractional form $m + n - \frac{2mn + n^2}{m - n}$.

485. A gentleman who had \$10,000, used a portion of it in building a house, and put the rest at interest for one year; $\frac{1}{3}$ of it at 6 per cent., and $\frac{2}{3}$ of it at 5 per cent. The income from both investments was \$320. What was the cost of the house?

486. There is a fraction such that if 3 be added to the numerator its value will be $\frac{1}{3}$, and if 1 be subtracted from the denominator its value will be $\frac{1}{5}$. What is the fraction?

487. Three boys, A., B., and C., had each a bag of nuts. Each boy gave to each of the others $\frac{1}{5}$ of the number he had in his bag. They then counted their nuts, and A. had 740, B. 580, C. 380. How many had each at first?

488. A. gives to B. and C. as much as each of them has; B. gives to A. and C. as much as each of them then has; and C. gives to A. and B. as much as each of them then has, after which each has \$8. How much had each at first?

489. Define the following: Simultaneous Equation; Involution; Evolution; Radical; Surd; Quadratic Equation, Incomplete and Complete; Ratio; Proportion.

$$490. \quad x + y + \frac{2}{z} = 3,$$

$$2x + 3y + \frac{1}{z} = 2,$$

$$3x - 2y + \frac{3}{z} = 11.$$

$$491. \quad x - ay + a^2z = a^3,$$

$$x - by + b^2z = b^3,$$

$$x - cy + c^2z = c^3.$$

492. A. and B. together do a piece of work in 15 days. After working together for 6 days, A. went away, and B. finished it by himself 24 days after. In what time could A. alone do the whole?

$$493. \quad \frac{3}{x-1} + \frac{4}{x-3} = \frac{15}{x+3}. \quad \text{Solve.}$$

$$494. \quad \frac{x-b}{x-a} - \frac{x-a}{x-b} = \frac{2(a-b)}{x-(a+b)}. \quad \text{Solve.}$$

495. Two trains run without stopping over the same 30 miles of rail. One of them travels 5 miles an hour faster than the other, and accomplishes the distance in 12 minutes less. Find the speed of the two trains.

496. The velocity of a falling body varies as the time during which it has fallen from rest. If the velocity of a falling ball at the end of 2 seconds is 64 feet, what will be its velocity at the end of 8 seconds?

497. The cost of sinking a well was \$45, \$1 being paid for sinking the first yard of depth, \$1.50 for the second, \$2 for the third, and so on. What was the depth of the well?

INDIANA.

Examination for Teacher's Certificate.

498. What is a linear equation? A quadratic equation? Illustrate each.

499. Represent graphically, with as much care as possible without the use of a ruler, the equation $x^2 + y^2 = 25$.

500. Divide 46 into two parts such that if one part be divided by 7 and the other by 3 the sum of the quotients shall be 10.

501. Divide (a) $\sqrt[4]{\frac{a}{b}}$ by $\sqrt[2]{\frac{a}{b}}$.

(b) $a^{\frac{2}{3}}$ by $a^{\frac{1}{2}}$.

502. Find the N^{th} power of $x \sqrt[m]{y}$.

503. What is an imaginary quantity? Illustrate.

504. Subtract $2\sqrt{-4}$ from $9\sqrt{-1}$.

505. Divide 100 into two such parts that the sum of their square roots shall be 14.

506. $a : b :: c : d$, prove that

(1) $a + b : b :: c + d : d$, and

(2) $a + b : a :: c + d : c$.

507. $\left. \begin{array}{l} 7x - 3y = 1 \\ 11z - 7u = 1 \\ 4z - 7y = 1 \\ 19x - 3u = 1 \end{array} \right\}$ Find values of x, y, z , and u .

NORTH CAROLINA.

Examination for Teacher's Certificate.

508. Tell why multiplying a "minus by a minus" gives a plus.

509. Simplify

$$b - \{b - (a + b) - [b - (b - \overline{a - b})] + 2a\}.$$

510. Square $y^2 - 6$, $4x^2 - 5y^2$, $ab - 2c^2$.

511. Write the products of $(x + 14)(x - 4)$,
 $(x - 1)(x + 8)$,
 $(x - 5)(x - 4)$.

Give rule for writing the products.

512. Find the value of y : $24 + 5y = 62$.

$$64 - 5y = 2.$$

513. If A. gives B. \$5 of his money, B. will have twice as much money as A. has left; but if B. gives A. \$5 of his money, A. will have three times as much as B. has left. How much money has each?

514. Find highest common divisor, or factor, of the following:

(a) $x^2 - 1$ and $x^2 - 2x + 1$.

(b) $x^2 + 2xy + y^2$ and $x^2 - y^2$.

(c) $m^3 + n^3$ and $m^2 - n^2$.

515. $\frac{6}{x} + \frac{8}{y} = 1.$

$$\frac{12}{x} + \frac{4}{y} = 1.$$

$$516. 2x - 3y = 1.$$

$$3y - 4z = 7.$$

$$4z - 5x = 32.$$

Find value of x , y , and z .

$$517. \text{ Find value of } x : \frac{x+4}{x-4} + \frac{x-4}{x+4} = \frac{10}{3}.$$

NEW HAMPSHIRE.

Examination for Teacher's Certificate.

518. Define and give an example of each of the following: Coefficient, Binomial, Reciprocal of a Quantity, Homogeneous Terms, and a Residual Quantity.

519. To what is each of the following expressions equal:
 x^{-3} ; $(a-b)(a+b)$; $(x-y)^3$; $a - [b - \{c - (d - e - f)\}]$;
 $\sqrt[3]{\frac{a}{3}} \sqrt{\frac{a}{3}}$?

520. (a) What is the apparent sign of a fraction? The real sign? What is an imaginary quantity?

(b) What is a quadratic equation? Give a method of completing the square in an affected quadratic equation.

521. (a) Factor $a^4 - b^4$; and $ab + ay + bx + xy$.

(b) If $xy = 50$ and $x^2 + y^2 = 125$, what is the value of x and of y ?

522. The head of a fish is 10 inches long; the tail is as long as the head and half the body; and the body is as long as the head and tail both. What is the length of the fish?

SOUTH DAKOTA.

Examination for Teacher's Certificate.

523. Divide 1 plus $\frac{a \text{ minus } 1}{a \text{ plus } 1}$ by $\frac{a \text{ minus } 1}{a \text{ plus } 1}$.

524. Define Quantity, Algebra.

525.
$$\begin{cases} 2x \text{ plus } 4y \text{ minus } 3z = 22. \\ 4x - 2y \text{ plus } 5z = 18. \\ 6x \text{ plus } 7y - z = 63. \end{cases}$$
 Find the value of x, y , and z .

526. A capitalist receives an annual income of \$2940. Four-fifths of his money bears an interest of 4 per cent., and the remainder yields 5 per cent. What amount has he invested?

527. Reduce $\sqrt[4]{a}$, $\sqrt[6]{5b}$, $\sqrt[8]{a^3 \text{ plus } b^2}$ to a common index.

528. In gunpowder there are equal parts of saltpeter and sulphur and four times as much charcoal as of both the others together. How much of each is there in ten pounds of powder?

KENTUCKY.

Examination for State Certificate.

529. Factor $x^6 - y^6$; $y^3 - 1$; $x^5 + y^5$.

530. Find G. C. D. of $2x^4 + 11x^3 - 13x^2 - 99x - 45$ and $2x^3 - 7x^2 - 46x - 21$.

531. Find the value of $\left\{ \frac{x}{1+x} + \frac{1-x}{x} \right\} \div \left\{ \frac{x}{1+x} - \frac{1-x}{x} \right\}$.

532. Given $\frac{1}{2} \left\{ \frac{2x}{3} + 4 \right\} - \frac{7\frac{1}{2} - x}{3} = \frac{x}{2} \left\{ \frac{6}{x} - 1 \right\}$ to find x .

533. A person bought a certain number of sheep for \$94; having lost seven of them, he sold one-fourth the remainder at prime cost for \$20. How many sheep had he at first?

534. If a carriage wheel $16\frac{1}{2}$ feet in circumference took one second more to revolve, the rate of the carriage per hour would be $1\frac{7}{8}$ miles less. At what rate is the carriage travelling?

535. Perform the operations indicated in the following: $\sqrt{128} - 2\sqrt{50} + \sqrt{72} - \sqrt{18}$; raise $(\sqrt{2} + \sqrt{3})$ to the fourth power; and find the value of x in the equation

$$\sqrt{2+x} + \sqrt{x} = \frac{4}{\sqrt{2+x}}.$$

536. Given $17x^2 + 19x = 1848$, to find the value of x .

537. A railway train ran one hour and met with an accident which delayed it 60 minutes, after which it continued the journey at $\frac{3}{4}$ of its former rate of speed, arriving at its destination 3 hours late. Had the accident occurred 50 miles further on, it would have arrived $1\frac{3}{4}$ hours earlier. What was the entire distance travelled?

538. A. can row a skiff 6 miles an hour with the current and 3 miles an hour against it; how far can he pass down the stream and yet return to the point from which he set out in 8 hours?

COLORADO.

Questions for the Examination of Applicants for
State Diplomas.

539. Factor the expression $x^4 + x^2y^2 + y^4$, and state principle involved in doing so.

540. Give definitions of the symbols a^0 ; $\sqrt[3]{a}$; $a^{\frac{2}{5}}$; a^{-n} .

541. Solve the equations $2x^4 - 3x^2 - 20 = 0$ —i. e., find the values of x , which satisfy the equation.

542. Describe method of solution of the system of equations $5x^2 - 4xy + y^2 = 1$
 $3x + y = 5$.

543. Given the following three terms of a geometric series :

1st term,	2d term,	3d term,
3 ,	$\frac{3}{2}$,	$\frac{3}{4}$

find the sum of 10 consecutive terms (commencing with 3 as first term) by using the general formula for the sum of geometric series.

544. The numerical value of the expression $\left(\frac{a\sqrt[3]{b}c^{\frac{2}{5}}}{d^5}\right)$ where a, b, c, d represent given positive numbers, is to be determined.

NASHVILLE, TENNESSEE.

Examination for City Schools.

545. If $a = 2$, $b = 5$, $m = 3$, $n = 1$, what will be the value of $5a^{m+2} + 15a^{m+1}b^n - 10a^mb^{n+2}$.

If $x = 8$, prove that $x - (\sqrt{x+1} + 2) - \frac{x - \sqrt[3]{x}}{\sqrt{x-4}} = 0$.

546. Subtract $(a - b)x - (b - c)y$ from $(a + b)x + (b + c)y$.
Simplify $10x - [3x - \{-4x - (-2x + 3x)\}]$.

547. The united ages of a father and son make 75 years. The father was 35 years old when the son was born. What is the age of each?

548. Multiply $a^2 - ab - ac + b^2 - bc + c^2$ by $a + b + c$.
Expand $(3x + 7)(3x - 7)$; $(4x^2 - 2x)^2$.

549. Multiply $3a^n - a^{n-1}b + 2a^{n-2}b^n$ by $2a - b$.

550. Solve $364 - 15(15x - 7) - 3(13 - 2x) + 8 = 6(x - 2)$.

551. Divide $m^3 + 3m^2n + 3mn^2 + n^3 - p^3$ by $m + n - p$.

552. Solve $(2x - a)(6x + a) - (4x + b)(3x - b) + 5ax = 0$.

553. Divide

$$a^{m+2} - a^{m+3} + 37a^{m+5} - 55a^{m+6} + 50a^{m+7} \text{ by } a^2 - 3a^3 + 10a^4.$$

554. If \$132 be divided among 5 men, 7 women, and 30 children, giving each man \$4 more than to each child, and to each woman 3 times as much as to each child, how much will each man, woman, and child receive?

555. Factor $a^2 + 2ad + 12bc - 9c^2 - 4b^2 + d^2$.

Factor $12x^2 - 10xy - 12y^2$.

556. Find H. C. D. of $6x^5 - 4x^4 - 11x^3 - 3x^2 - 3x - 1$, and $4x^4 + 2x^3 - 18x^2 + 3x - 5$.

557. Simplify $\frac{\frac{m^2 + n^2}{n} - m}{\frac{1}{n} + \frac{1}{m}} \times \frac{m^2 - n^2}{m^3 + n^3}$.

558. A person walks to the top of a mountain at the rate of $2\frac{1}{3}$ miles per hour, and down the same way at the rate of $3\frac{1}{2}$ miles per hour, and is out 10 hours. How far is it to the top of the mountain?

559. Solve $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 3$; $\frac{a}{x} + \frac{b}{y} - \frac{c}{z} = 1$; and

$$\frac{2a}{x} - \frac{b}{y} - \frac{c}{z} = 0.$$

560. A number is expressed by three digits. The sum of the digits is 21; the sum of the first and second exceeds the third by 3; and if 198 be added to the number the digits in the units' and hundreds' places will be interchanged. Find the number.

561. Simplify :

$$(a) \left(\frac{a^{-\frac{1}{2}} b^{\frac{1}{2x}}}{x^{\frac{1}{n}} y^{-\frac{1}{2n}}} \right)^{-2n} \cdot (b) \sqrt[n]{a^{\frac{1}{n}} b^{-n^2} x^{-3n}}.$$

562. Write the first four terms of $\left(2 - \frac{x}{a}\right)^3$.

563. Find the square root of $x^4 - \frac{2x}{3} + x^2 + \frac{1}{9} + 2x^3 - \frac{2x^2}{3}$.

564. Find the cube root of $x^6 - 3ax^5 + 5a^3x^3 - 3a^5x - a^6$.

565. Find the square root of $x^{-4} + 4x^{-3} - \frac{2}{3}x^{-\frac{3}{2}} + 4x^{-2} - \frac{4}{3}x^{-\frac{1}{2}} + \frac{1}{9}x$.

566. Solve $x^2 - 2ax - bx + a^2 + ab - 2b^2 = 0$.

567. Solve $\left. \begin{aligned} x^2 + xy + y^2 &= 37 \\ x^4 + x^2y^2 + y^4 &= 481 \end{aligned} \right\}$.

568. A cyclist coasting on a steep grade moves $1\frac{1}{2}$ feet the first second, $4\frac{1}{2}$ feet the next second, $7\frac{1}{2}$ feet the third, and so on; if he reaches level ground in 10 seconds, how long is the incline?

569. Develop a formula for 's' having r , n , and l given.

570. Divide $6 - 3\sqrt{-3} - 2\sqrt{-2} - \sqrt{6}$ by $2 - \sqrt{-3}$.

571. (a) In a recent election for "Maids of Honor" there were 15 names voted for. If only two names could appear on the ballot, how many different ballots could be prepared?

(b) A child has six blocks, each a different color; in how many different ways can he place them, using four at a time?

572. Find the seventh term of $\left(\frac{x}{2} + \frac{y}{3}\right)^{10}$.

573. Prove $a^{-n} = \frac{1}{a^n}$.

574. Solve for x and y , $\frac{1}{x} + \frac{1}{y} = \frac{9}{20}$, $\frac{1}{x^2} + \frac{1}{y^2} = \frac{41}{400}$.

575. The sum of a geometric series is 1. The first term is $\frac{1}{3}$, the last term is 0. What is the ratio? Number of terms?

576. Divide $\sqrt{3} + \sqrt{-5}$ by $-\sqrt{-5} - \sqrt{-3}$.

577. Solve $\sqrt{2 - \sqrt{4 - 1}}$.

578. Solve $\frac{2}{x} + \frac{3}{y} = 3$.

$$\frac{5}{x} + \frac{2}{y} = 2.$$

579. A man is told to select 4 books from a pile of 12 different books. How many selections might he make?

580. Solve $\frac{x^2 + 3x + 2}{(x + 1)^3} = 1$.

NEBRASKA.

Examination for State Certificate.

581. Extract the sixth root of $1 + 12x + 60x^2 + 160x^3 + 240x^4 + 192x^5 + 64x^6$.

582. A. rode a certain distance at the rate of m miles an hour, and walked back at the rate of n miles an hour, and performed the whole journey in t hours. How far did he ride?

583. $\sqrt{x} - 3 = \frac{2\sqrt{10}}{\sqrt{x+3}}$. Find x .

584. Define Symbols, Signs, Terms, Positive, Negative, Coefficient.

585. Factor the following :

(a) $x^2 - 8x + 7$.

(b) $x^2 - x - 20$.

(c) $x^2 + 9x - 36$.

(d) $x^2 - y^2 - 2yz - z^2$.

586. Determine the values of x , y , and z :

$$\begin{cases} 2x + y + 3z = 19. \\ 3x + 2y - z = 8. \\ x - 3y + 2z = 1. \end{cases}$$

587. If a certain rectangle were 1 foot longer and 1 foot broader, it would contain 14 square feet more area. If it were 1 foot shorter and $\frac{1}{2}$ foot wider, its total area would be unchanged. What are its dimensions?

588. Find the G. C. D. of $2x^3 + 4x^2 - 7x - 14$ and $6x^3 - 10x^2 - 21x + 35$.

589. Solve $\frac{1}{x-2} - \frac{1}{x-4} = \frac{1}{x-6} - \frac{1}{x-8}$.

590. Divide $a^{2m \text{ plus } n} + a^n b^{2m} - a^{2m} b^n - b^{2m \text{ plus } n}$ by $a^n - b^n$.

UTAH.

Examination for Teacher's Certificate.

591. Find the L. C. Dd. of $4a^2 - 6ab - 18b^2$, $4a^3 + 4a^2b - 3ab^2$, and $6a^4 + 5a^3b - 6a^2b^2$.

592. A certain sum of money, at simple interest, amounted in m years to a dollars, and in n years to b dollars. Required the sum, and the rate of interest.

593. Solve the following equations :

$$(a) (2x + 5)^{-5} + 31 (2x + 5)^{-\frac{5}{2}} = 32.$$

$$(b) \sqrt{x^2 + 2x + 9} = x^2 + 2x + 3.$$

594. Solve the following equations : $x - \frac{2ab}{y} = a.$

$$y - \frac{2ab}{x} = b.$$

595. A rectangular garden is surrounded by a walk 7 feet wide ; the area of the garden is 15,000 square feet, and of the walk 3696 square feet. Find the length and breadth of the garden.

596. Define *imaginary quantity*, and perform the following operations as indicated :

Multiply $4 + 3\sqrt{-2}$ and $2 - 5 - \sqrt{-2}.$

Divide $\sqrt{-294}$ by $\sqrt{-3}.$

597. (a) Find a fourth proportional to $\frac{2}{3}$, $\frac{3}{5}$, and $\frac{2}{7}.$ (b) Find a third proportional to $\frac{3}{4}$ and $\frac{5}{6}.$

598. Find four integers in arithmetic progression such that their sum shall be 24, and their product 945.

599. Reduce to simplest form :

$$\frac{\frac{x}{1+x} + \frac{1-x}{x}}{\frac{x}{1+x} - \frac{1-x}{x}}; \text{ also } \frac{2x-1-\frac{x-1}{2-\frac{x}{x-1}}}{x-\frac{x}{1+x}}$$

600. Find the L. C. Dd. of $x^2 + 4xy + 4y^2$, $x^2 + xy - 2y^2$, $x^3 + 8y^3$.

Find L. C. Dd. of $12m^2 - 21m - 45$, $4m^3 - 11m^2 - 6m + 9$.

601. Find H. C. F. of $x^2 - 6x - 27$, $x^2 + 6x + 9$, $x^3 + 27$.

Find H. C. F. of $3a^4 + 5a^3 + 12a^2 + 8$, $6a^4 - 10a^3 + 19a^2 - 10a - 4$.

602. Write the fifth term of the expansion of $(x-y)^n$. Expand $(a+b)^7$ by the binomial theorem—leaving all your work.

603. Find the values of x and y from these equations :

$$\frac{x+y}{x-y} = \frac{1}{m-n}; \quad \frac{x+m}{y+m} = \frac{1+m}{1+n}$$

604. Prove $x^0 = 1$; also $a^{-n} = \frac{1}{a^n}$.

605. Express with positive exponents and simplify :

$$\left(\frac{y^{-3}}{x^{\frac{2}{7}}z^{-1}}\right)^{-\frac{3}{2}} \times \left(\frac{y^{\frac{14}{3}}x}{z^{-\frac{21}{4}}}\right)^{-\frac{2}{7}}$$

606. Prove : If four quantities are in proportion, they are in proportion by composition.

Note.—Verification is not proof.

607. A man bought 6 ducks and 2 turkeys for \$15. For \$14 he could buy 4 ducks more than he could turkeys for \$9. Find the price of each.

608. A farmer having sold, at \$75 each, horses which cost him x dollars apiece, finds that he has realized x per cent. profit on his outlay. Find x .

ANSWERS.

Page 15.

- | | |
|--------|---------|
| 1. 54. | 6. 119. |
| 2. 4. | 7. 1. |
| 3. 1. | 8. 79. |
| 4. 8. | 9. 202. |
| 5. 48. | 10. 0. |

Page 16.

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|---------|------------|
| 1. 54. | 6. 186. |
| 2. 0. | 7. 129. |
| 3. 32. | 8. 60. |
| 4. 98. | 9. 36. |
| 5. 146. | * 10. 263. |

Page 16.

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|--------|-----------|
| 2. 55. | 4. \$.25. |
| 3. 2. | |

Page 17.

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| 5. \$.01. | 11. 146. |
| 6. 27. | 12. \$82. |
| 7. 163,504. | 13. \$264. |
| 8. 2. | 14. \$40. |
| 9. \$600. | 15. \$.52. |
| 10. \$169. | |

Page 18.

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|------------------------|
| 2. 5. |
| 3. 36. |
| 4. 0. |
| 5. 100, 0, 2500, 1, 3. |
| 6. 0. |

7. 50.

9. 0.

8. 0.

10. 20.

Page 19.

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|---------|--------|
| 11. 68. | 12. 6. |
|---------|--------|

Page 20.

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|-------------|--------------|
| 1. $x=5$. | 9. $x=7$. |
| 2. $x=5$. | 10. $x=3$. |
| 3. $x=6$. | 11. $x=9$. |
| 4. $x=11$. | 12. $x=1$. |
| 5. $x=-3$. | 13. $x=9$. |
| 6. $x=-6$. | 14. $x=60$. |
| 7. $x=-2$. | 15. $x=12$. |
| 8. $x=5$. | |

Page 21.

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|-------------|---------------|
| 2. $3x$. | 6. 13, 39. |
| 3. $4a$. | 7. 24, 96. |
| 4. 20, 40. | 8. 6, 18, 36. |
| 5. 50, 250. | |

Page 22.

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| 9. 5, 30. | 10. 10, 30, 90. |
|-----------|-----------------|

Page 23.

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|--------------|--------------------|
| 15. 22. | 18. 5, 6, 7. |
| 16. 50, 100. | 19. 100, 200, 200. |
| 17. 18, 30. | 20. 0. |

Page 24.

22. 10, 120. 25. 45.
 23. 49. 26. 24, 18.
 24. 12, 48.

Page 26.

5. $-2x$. 9. $3ax$.
 6. $-4xy$. 10. $2a + 2$.
 7. $x^2 - 7x$. 11. $4x^3 - 8$.
 8. 0.

Page 27.

12. $4x^3 - 8$.
 13. $2x^3 - 4$.
 14. $-1 - 2x + 2x^2 + 2x^3 + 3x^4$.
 15. $4x^2 - z^2$.
 16. $8x$.
 17. $-8m$.
 18. $2x + 3y$.
 19. $-3x$.
 20. $+3x$.
 21. 1. $16a - 10b - 11c$.
 2. $11x^2 - 6x - 5$.
 3. $6x^2 + 4ax$.
 4. $7a^2 - 10ab + 3b^2$.
 5. $-8x^2y^3 - 5x^2y^2 + 3x^3y^2$.
 22. $7a - 10b$.
 23. $-x^2 + 4y^2 + 3z^2$.
 24. $2b$ and $-2b$.
 25. $x^3 + x$.

Page 28.

1. $9x - 13y$.
 2. $-a + b + 3c$.
 3. $-x^2 - xy + y^2$.
 4. $a^2 - 3b^2 + 3c^2$.
 5. $6x - 5y$.
 6. $2m - n - p$.
 7. $1 - 2x$.
 8. $x - y$.
 9. -1 .

10. $25 - 2x$.
 11. $a^2 + a + 6$.
 12. $a - b - d + 2c$.
 13. a .

Page 29.

14. $16 - 12x$. 15. $x + 3a$.
 3. 0. 7. 5.
 4. 15. 8. -2 .
 5. -5 . 9. 4.
 6. 4. 10. 6.

Page 30.

1. $2a - 2b - 2c - (-d - e + f)$.
 2. $3x - 2a - y - (4b + z + c)$.
 3. $a - b + c - (5a + b - 2)$.
 4. $by + ax + bx - (-cx - cz + xy)$.
 5. $x + y - z - (-u + 4 - 5a)$.

REVIEW.

1. 1. $x = 5$. 4. $x = -70$.
 2. $x = 12$. 5. $x = 0$.
 3. $x = 20$.
 2. 1. $6x^3 - 4a^2 + 10a^3$.
 2. $4a + 3b - 3c - d$.

Page 31.

3. $8a^2 + ab - 5b^2 - 3c^2$.
 4. $-2a + ab - 3f$.
 5. $6a - 5b + 2c - 5ac$.
 3. 1. $3ac - 8ab + 7c + c^2$.
 2. $5a^3 + 2a^2 - 7a^2b + 11b^2c$.
 3. $-a^3 + 2a^2 + a + 2$.
 4. $-2x^2 - 2ax$.
 5. $-4a^3 + 5a^2b - 5ab^2 + b^3$.
 4. 1. $2x + 4$.
 2. $2a - b - d$.
 3. $9x$.
 4. $a^2 - 3b^2 + 3c^2$.
 5. $6x - 5y$.

5. 1. 21. 4. -3000 .
 2. 5. 5. -9 .
 3. 40.

6. 1. $6a + 9b - 6c + (4d - 3e + 2f)$.
 2. $b - (x + y - 3z + 2b - c)$.

Page 32.

3. $a^6 + 4a^5 - (2a^3 + 4a^2 + a + 1)$.
 4. $-3a - 2b + (6c - 4d - e - 2f)$.
 5. $ax - cy - (dz - bz - cd - ax)$.

Problems.

1. $-18a - 8b - 10c + 13x$.
 2. $2a + 7b - 8c$.
 3. $2ab + 2b^2$.
 4. $-b^2 - 2ab$.
 5. $-a^2 - 2ab - b^2$.
 6. $-x^2 + y^2$.
 7. $3ax$.
 8. $x - p + p = 0 + p$.
 $\therefore x = p$.
 9. $-11a^2b + 4ab^2 - 12a^2b^2 - b^3$.
 10. $x + 9$.
 11. $x + y + 2$.

Page 33.

12. $\frac{1}{5}$.
 13. 1. $-3a^2 + 3a^2b + ac + 7by$.
 2. $3a^2 + 9a^2b + 21by - ac$.
 3. $-a^2 - 9ac - 3a^2b - 7by$.

Page 35.

1. 90. 10. $35bc^2dx^2$.
 2. $24x$. 11. $-8f^3g^4$.
 3. $-60xy$. 12. $27x^3y^3z^3$.
 4. 0. 13. $-132e^3f^2x$.
 5. $-70a^2y^3$ 14. $-48a^{2n}$.
 6. $-72a^2$. 15. $-30x^{2n}y^{2m}$.
 7. $-48a^2y^2$. 16. $-36y^{2n}$.
 8. $108a^2x^4$. 17. a^{5n} .
 9. $-56axy^2$. 18. a^2y^{4n} .
 20. 1. $-9abx^2y + 6abxy^2$.

Page 36.

2. $14ax^2y - 8bxy^2$.
 3. $-24x^3y^2 + 6x^2y^3$.
 4. $16anxc^2 - 16an^2m^2$.
 5. $-9m^3n - 9m^4n + 9m^5n$.
 6. $3x^{n+2} - 4x^{n+1}$.
 7. $12a^{3n}y^m - 24a^{2n}ym+1$.
 8. $35x - 25x^2$.
 9. $15x^3 - 10x^2y - 20x^2$.
 10. $-3x^3 + 9x^2y - 9xy^2 + 3y^3$.
 21. 1. $25m^2n^2$. 4. $3m^5n^3p^6$.
 2. $9a^3b^3$. 5. $-1^2_5x^2y^2a^2c$.
 3. $a^5b^4c^4$.

Page 37.

2. 1. $6x^2 - 11x - 10$.
 2. $8x^2 - 10x - 12$.
 3. $6x^2 - 12x - 48$.
 4. $20x^2 - 41xy + 20y^2$.
 5. $40x^4 + 2x^2y^2 - 24y^4$.
 6. $42x^2y^2 + xy - 56$.
 7. $45a^5d + 6a^3b^2d^2 - 3ab^4d^3$.
 8. $48x^6y + 4x^4y^3 - 24x^2y^5$.
 9. $a^2x + 2abx + a^2y + 2aby$.
 10. $ax^3 + 2ax^2y - bx^3 - 2bx^2y$.
 2. 1. $m^3 - n^3$.
 2. $6a^3 + 23a^2b + 22ab^2 + 5b^3$.

Page 38.

3. $x^6 + 8ax^6 + 8a^2x^6 + a^3x^6$.
 4. $x^4 + abx^3 - abxy^2 - y^4$.
 5. $x^4 - y^4$.
 6. $x^4 + x^2y^2 + y^4$.
 7. $x^4 - 1$.
 8. $-x^8 + x^4 + 2x^2 + 1$.
 9. $2x^6 - 10x^5 + 5x^4 - 22x^3 - 5x^2 + 5x + 1$.
 10. $x^8 + 2x^6 + 3x^4 + 2x^2 + 1$.

Page 39.

2. $a^2 - 2b^2$. 4. 0.
 3. b^2 . 5. $2a^3 + 2b^3$.

1. $x^2 + y^2$.
2. $-2ab$.
3. 0
4. $-6a$.
5. $x^6 - y^6$.

REVIEW.

1. 1. $88x^{2m}$.
2. $-64x^{m+n}y^{2n+2}z^4$.
3. $m^6n^3 + m^3n^4 + m^3n^5$.
4. $a^6b^3 - 3a^5b^4 + 3a^4b^5 - a^3b^6$.

Page 40.

5. $a^2 - b^2 + 2bc - c^2$.
6. $12x^{2m+1}y^3 - 24x^{m+6}y^{n+3} + 12x^5y^{2n+3}$.
7. $9a^4 - 13a^2 + 4$.
8. $m^4 - 5m^2 + 4$.
9. $6(x+y)^{2n+3} + 4(x+y)^{2n+2} - 2(x+y)^{2n+1}$.
10. $-15a^4 + 37a^2bd - 29a^2cf - 20b^2d^2 + 44bcd^2f - 8c^2f^2$.
11. $x^6 + x^5 + x^3 + x - 1$.
12. $1 - 2x - 31x^2 + 72x^3 - 30x^4$.
13. $a^3 + b^3 + c^3 - 3abc$.
14. $2a^{2m} + 2a^mb^p - 4a^mc^n - 3a^mb - 3b^{p+1} + 6bc^n$.
2. 1. $a^2 + 2ab + b^2$.
2. $a^2 - 2ab + b^2$.
3. $a^2 - b^2$.
4. $a^3 - b^3$.
5. $a^3 + b^3$.
6. $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.
7. $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$.
8. $(a+1)^3 = a^3 + 3a^2 + 3a + 1$.
9. $(a-1)^3 = a^3 - 3a^2 + 3a - 1$.
10. $(a+b)^2(a-b) = a^3 + a^2b - ab^2 - b^3$.
11. $a^3 - a^2b - ab^2 + b^3$.

12. $a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$.
13. $a^2 + b^2 + c^2 - 2ab - 2ac + 2bc$.

3. 1. $x^2 - y^2$.
2. $x^2 + 2xy + y^2$.
3. $x^2 - 2xy + y^2$.
4. $x^3 + 3x^2y + 3xy^2 + y^3$.
5. $x^3 - 3x^2y + 3xy^2 - y^3$.
6. $x^3 - x^2y - xy^2 + y^3$.
7. $x^2 + y^2 + z^2 + 2xy + 2xz + 2yz$.
8. $x^2 + y^2 + z^2 - 2xy - 2xz + 2yz$.

Page 41.

4. 1. 1. 4. 6.
2. 1. 5. -9.
3. -2.
5. 1. 0.
2. $8b^3$.
3. $2b$.
4. $(m+1)^{4b}$.
5. $x^3 - 4bx^2 + 5b^2x - 2b^3$.

Page 42.

1. 1. $4a^2$. 6. -7.
2. $-6x^2$. 7. $-4a^6b^2$.
3. $-10y^2$. 8. $5z^4b^2$.
4. $-3z^2$. 9. $6m^4n^2$.
5. $-8x^7y$. 10. $-5pq$.
2. 1. $1 + 2a^2$.
2. $3a^2 - 4$.
3. $3 - 2a^2$.
4. $3x + xy$.
5. $x^2z^2 + 2$.
6. $-a + b + c$.
7. $-2a^2 + 4ab + 6b^2$.
8. $x^3 - xy - 1$.
9. $-4x^2 + 2xy + 3y^2$.
10. $2m^2 - 4mn - 10n^2$.
3. 1. $1 - m - m^2 + m^3$.
2. $0 + 1 + x - x^2$.

Page 43.

3. $1 - a + a^2$.
4. $3b^0c^0 - 2bc$.
5. $a^{-2} + a^{-1}b - 2c - a^2b^{-1}$.
6. $1 - 16 + 27 - 16 + 5 = 1$.
7. $x^{-1} + x^{-2} + x^{-3} + x^{-4}$.
8. $13x^{m+1} + 7x^n - x^{n-2}$.
9. $37a^{3n} + 21a^{2n} - 1$.
10. $3z^{r-1} - 7x^2y^n$.

Page 45.

1. $2x + 3$.
2. $2x - 5$.
3. $x - 3y$.
4. $a + x$.
5. $a^2 - 2ay + y^2$.
6. $2x^3 + 4x^2 + 8x + 16$.
7. $a - b - c$.
8. $2x - 3a$.
9. $36x^3 - 27x^2y - 16xy^2 + 12y^3$.
10. $x^4 - x^2y^2 + y^4$.
11. $x^2 + ax + a^2$.
12. $a^2 + 2ab + a$.
13. $x^4 + 2x^3 + x^2 - 4x - 11$.
14. $x^2 + y^2 + z^2 - xy - xz - yz$.
15. $x^3 - 2x^2 - x + 1$.
16. $x^3 + x^2y + xy^2 + y^3$.
17. $x^n - y^n$.
18. $x^3 - x^{-3}$.
19. $1 + x^2 + x^4 + x^6$.
20. $9a^2 + 4b^2 + 9c^2 - 6bc + 9ac + 6ab$.

REVIEW.

1. 1. $a + b + c$.
2. $b + m + 3ab - c + 7am$.
3. $0(a + x)$, or 0.
4. $4\sqrt{x}$.
5. $(2a + c)\sqrt{x}$.
2. 1. $-6a^2x + 7a^2$.

Page 46.

2. $10a^3b^2c + 9abc$.
3. $19a^m - 17b^n - 2c^p + 10d^q$.
4. $-a + 112 - \frac{1}{4}c + \frac{7}{2}bc$.
5. $-\frac{5}{2}y - \frac{5}{12}a - \frac{1}{12}x$.
3. 1. $8 + 3x$. 4. $9x$.
2. $4b$. 5. $6x - 5y$.
3. $2 - 5x$.
4. 1. $x + a$.
2. $x - a$.
3. $x - a$.
4. $x + a$.
5. $x^2 - ax + a^2$.
6. $x^2 + ax + a^2$.
7. $x^3 - ax^2 + a^2x - a^2$.
8. $x^3 + ax^2 + a^2x + a^3$.
9. $x^2 - ax + a^2$.
10. $x^2 + ax + a^2$.
11. $x^2 + y^2 + z^2 - yz - zx - xy$.
12. $4x^2 + 2xy + y^2$.
5. 1. $x^3 - 15x^2 + 71x - 105$.
2. $8x^3 + 30x^2y - 53xy^2 - 15y^3$.
3. $x^8 - 1$.

Page 47.

4. $x^8 - 256y^8$.
5. $x^8 - 2x^4y^4 + y^8$.
6. $x^{12} - 3x^8 + 3x^4 - 1$.
7. $a^4 + 4a^2x^2 + 16x^4$.
8. $81a^4 + 9a^2x^2 + x^4$.
9. $2a^{2m} + 2a^mb^p - 4a^mc^n - 3a^mb - 3b^{p+1} + 6bc^n$.
10. $2a^{4n} - 2x^{4n}$.
11. $\frac{4}{9}x^4 - \frac{1}{16}y^4$.
12. $1 - x^8$.
6. 1. $3.5x^2 - 3x + 1.5$.
2. $4x^3 - 3x^2 + .5x$.
3. $\frac{1}{6}d^3 - \frac{2}{3}d^2 + \frac{1}{2}d$.
4. $x^2 + 2x + \frac{1}{3}$.
5. $2a^3 - 2.49a^2x - 3ax^2 + .5x^3$.

Page 49.

1. $b^2 + 2bc + c^2$.
2. $b^2 - 2bc + c^2$.
3. $a^2 + 2ax + x^2$.
4. $a^2 + 4ab + 4b^2$.
5. $9x^2 - 12dx + 4d^2$.
6. $16m^2 - 24mn + 9n^2$.
7. $4a^4 + 12a^2x^2 + 9x^4$.
8. $x^2 - 10x + 25$.
9. $9a^2 + 30a + 25$.
10. $49 - 28x^2 + 4x^4$.
11. $b^2 - d^2$.
12. $16x^2 - 9$.
13. $4a^4 - 4b^4$.
14. $16a^2 - c^2$.
15. $a^2 - 49b^4$.

Page 50.

2. $(a + 3b)(a + 3b)$.
3. $= x^2(1 - y)^2$.
4. $(1 + x)(1 - x)$.
5. $(5 + 4a)(5 - 4a)$.
6. $(2x - 3a)(2x - 3a)$.
7. $(a + 5b)(a + 5b)$.
8. $(x - 8)(x - 8)$.
9. $9(2m - n)^2$.
10. $x^4(x + 4)^2$.
11. $(3x + 4y)(3x - 4y)$.
12. $(5a + b^2)(5a - b^2)$.
13. $(7m + 10n^3)(7m - 10n^3)$.
14. $9(4ab^2 + 5c^3)(4ab^2 - 5c^3)$.
15. $a^2b^2(2a + 13b)^2$.
1. $(x - y + z)(x - y - z)$.
2. $(2y + 2z - x)(2y - 2z + x)$.
3. $8(a + b)(a - b)(a^2 + b^2)$.
4. $a^2(a^2 - 4bc)$.
5. $(m + x - y)(m - x + y)$.
6. $(a + b + c + d)(a + b - c - d)$.
7. $(3a + 5 + 2a - 3)(3a + 5 - 2a + 3) = (5a + 2)(a + 8)$.

8. $(x + 2)x$.
9. $(a - b + c - d)(a - b - c + d)$.
10. $(9x + 7y)(5x + y)$.

Page 51.

1. $(a + d + 1)(a - d + 1)$.
2. $(x + q - 1)(x - q - 1)$.
3. $(a - b + c)(a - b - c)$.
4. $(x - b + a)(x - b - a)$.
5. $(c + d + 1)(c + d - 1)$.
6. $(a - m + b + n)(a - m - b - n)$.
7. $(a + b + m - n)(a - b + m + n)$.
8. $(a - d + b - c)(a - d - b + c)$.
9. $(3 - x - a)(a - x - 1)$.
10. $(a + x)^2(1 - 4b^4)$.

Page 52.

1. $x(x + y + 2)(x + y - 2)$.
2. $y(a - b + c)(a - b - c)$.
3. $m(2m^2 + 2b - 1)(2m^2 - 2b + 1)$.
4. $5(a - m + n + b)(a - m - n - b)$.
1. $(2x + y)(3x - a)$.
2. $(x + 1)(x^3 + 2)$.
3. $(x - 2y)(m - n)$.
4. $(y - 1)(y^2 + 1)$.
5. $q^2(a + c + b + d)(a + c - b - d)$.
6. $xy(x + y)(x - y)^2$.

Page 53.

5. $(ax + by)(mx - ny)$.
6. $(a + x)(a - x)(ax - m)$.
7. $(2x^2 - a)(2x - 1)$.
8. $(y + 10)(y - 11)$.
9. $(x + y)(1 - x - y)$.
10. $(c + x)2c$.
1. $(x^2 + x + 1)(x^2 - x + 1)$.
2. $(x^2 + a^2 + xa)(x^2 + a^2 - xa)$.
3. $(2a^2 + 3a - 1)(2a^2 - 3a - 1)$.

4. $(3a^2 + 3a - 1)(3a^2 - 3a - 1)$.
5. $(7b^2 + 5d^2 + 9bd)(7b^2 + 5d^2 - 9bd)$.
6. $(3x^2 + 2b^2 + 4bx)(3x^2 + 2b^2 - 4bx)$.
7. $(5x^2 - 2y^2 + 4xy)(5x^2 - 2y^2 - 4xy)$.
8. $(7a^2 - 12y^2 + ay)(7a^2 - 12y^2 - ay)$.
9. $(15a^2 + 7b^2 + 16ab)(15a^2 + 7b^2 - 16ab)$.
10. $(16c^2 + 5d^2 + 14cd)(16c^2 + 5d^2 - 14cd)$.
5. $(2a - b)(4a^2 + 2ab + b^2)$.
6. $(x^3 + y^3)(x^3 - y^3) = (x + y)(x - xy + y^2)(x - y)(x^2 + xy + y^2)$.
7. $(1 + x^4)(1 + x^2)(1 + x)(1 - x)$.
8. $(a - b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4)$.
9. $(x^4 + y^4)(x^4 - y^4) = (x^4 + y^4)(x^2 + y^2)(x + y)(x - y)$.
10. $(m + n + 2)(m^2 + 2mn + n^2 - 2m - 2n + 4)$.

Page 59.

1. $(x + 3)(x + 3)$.
2. $(x + 1)(x + 2)$.
3. $(x + 7)(x + 10)$.
4. $(x + 1)(x - 2)$.
5. $(x + 2a)(x + 3a)$.
6. $(x^2 + 2)(x + 3)(x - 3)$.
7. $(x^2 + 4y)(x^2 - 3y)$.
8. $a^4(a - 5)(a - 7)$.
9. $(x - 10)(x - 20)$.
10. $3ab(a + 3)(a + 3)$.
1. $(x^2 + y^2)(x^2 - y^2 + 1)$.
2. $(x - a)(x^2 + ax + a^2 + 1)$.
3. $(2x - y)(4x^2 + 2xy + y^2 - 1)$.
4. $(a + 2b)(a^2 - 2ab + 4b^2 + a + 3b)$.
5. $(6 - a)(42 + 7a + a^2) + 6$.
6. $(m + n)^2(m + n)(m - n)$.
7. $(a^2 + b^2)(a^2 - b^2)(a^4 + b^4 - 1)$.
8. $9(a - y)(3a^2 + 3ay + 3y^2 - a + y)$.
9. $(x - 2y)(x^2 + 2xy + 4y^2)(p + 2q)(p - 2q)$.
10. $(a + 2)(a - 2)(a - 1)(a^2 - 2a + 4)(a^2 + 2a + 4)(a^2 + a + 1)$.

Page 57.

1. $12(x + 1)(x + 1)$.
2. $3(2x + 5)(x + 2)$.
3. $(2x - 3)(3x + 4)$.
4. $2(x - 3)(2x + 1)$.
5. $(5x - 1)(3x + 7)$.
6. $(x - 11)(5x - 2)$.
7. $(5 - 2x)(3x + 4)$.
8. $(a + b + 2)(a - b - 3)$.
9. $(ax + b)(y + c)$.
10. $(a^2 + 3)(a^2 - 3)(b^3 + 10)$.

Page 58.

1. $(m + n)(m^2 - mn + n^2)$.
2. $(a + 4)(a^2 - 4a + 16)$.
3. $(xy - 4)(x^2y^2 + 4xy + 16)$.
4. $(7 - x)(49 + 7x + x^2)$.

Page 60.

1. $(4a + b + c)(a - 3b + 4c)$.
2. $(x + a - 3c)(2x - a + 7c)$.
3. $(x + 2y - 3c)(3x - y + 2c)$.
4. $(a + 3b + c)(4a - 2b - c)$.
5. $(4x - y + z)(2x + 2y + 2z)$.
6. $(a - b - c)(2a - 2b - 2c)$.
7. $(2a + 4x - 6)(a + x - 6)$.
8. $(a - 3b + 4c)(4a + b + c)$.
9. $(x + 2y - 3c)(3x - y + 2c)$.

Page 61.

1. $(x + 1)(x^2 - x - 2)$.
2. $x(x + 1)(x^2 - x - 2)$.

3. $(a+2)(a+2)(a-1)$.
4. $(x+1)(x+2)(x-3)$.
5. $(x+1)(x+2)(3x-2)$.
6. $(x-4)(x+3)(x-2)$.
7. $(x-2)(x-5)(x-1)$.
8. $(2x-1)(3x-1)(x-3)$.
9. $(2x+5)(2x-3)(x-1)$.
10. $(x+3)(2x+5)(x-1)$.

Page 62.

1. 2, 3, a , a , a , b , b , c .
2. $a(x-1)$.
3. $3a(a-2b)$.
4. $3m^2(m-4)$.
5. $(a+b)(a+b)$.
6. $(3x^3+y)(3x^3+y)$.
7. $5a^3(3x-2y+z)$.
8. $a(a+3)(a-3)$.
9. $(a^2+4b^2)(a+2b)(a-2b)$.
10. $(y+5x)(y-5x)$.
11. $(6x+5b)(6x-5b)$.
12. $a(a^2-ay+y^2)$.
13. $(a-4)(a+2)$.
14. $(m+n+1)(m+n-1)$.
15. $(x^2+xy+y^2)(x^2-xy+y^2)$.
16. $(x-2)(x+2)(x^2+2x+4)$
 (x^2-2x+4) .
17. $(x+2yz)(x^2-2xyz+4y^2z^2)$.
18. $(a-b)(a+b+1)$.
19. $(x+1)(x-1)^3$.
20. $(x-2y)(x+15y)$.
21. $(3a-2b)(9a^2+6ab+4b^2)$.
22. $(3x+4y)(9x^2-12xy+16y^2)$.
23. $(x^2+x-2)(x^2-x+2) =$
 $(x+2)(x-1)(x^2-x+2)$.
24. $(x-3)(x-7)(x+1)$.
25. $xy(x+y)(x-y)(x-y)$.
26. $(a-20b)(a+2b)$.
27. $(a+1)(a-1)(a^2-6)$.
28. $(x+y)2y$.
29. $(x-12)(x+5)$.
30. $(a+b)(x-3)(x-2)$.

31. $5x^2(x-6)(x+3)$.
32. $7a^2x(2x-a)(x-2a)$.
33. $(9+x)(3+x)$.
34. $a^2(3a-b)(a+b)(3a+b)$
 $(a-b)$.
35. $(3x^2+3xy+5y^2)(3x^2-3xy+5y^2)$.
36. $4abxy$.
37. $(a-b+c)(a-b-c)$.
38. $5x^2y(4y^2+x^2)(2y+x)$
 $(2y-x)$.
39. $(x^2+x+1)(x^2-x+1)(x^4-x^2+1)(x^8-x^4+1)$.
40. $(5a^2b^2+7c^2)^2$.
41. $(a \pm b \mp c)(a \pm b \pm c)$.
42. $(x+a)(x-a)(x-a)$.
43. $(x-2)(x^2-x+1)$.

Page 63.

44. $(a^2+ab+b^2)(a-b+c)$.
45. $(m+n)(m^6-m^5n+m^4n^2-m^3n^3+m^2n^4-mn^5+n^6)$.
46. $(a^2b^3+c^4)(a^2b^3-c^4)$.
47. $(1-x)(1-x)(1+x+x^2)$.
48. $(10xy^2+9z)(10xy^2-9z)$.
49. $(x-b+a)(x-b-a)$.
50. $(a+b-c)(a-b+c)$.
51. $(x+5)(x-5)(x+2)(x-2)$.
52. $(a+1)(a^2-a+1)(a-1)$
 (a^2+a+1) .
53. $(a+b)(a^2-ab+b^2+1)$.
54. $(a+3b)(a-3b+1)$.
55. $x^3(x-12)(x-11)$.
56. $(x-a)(x+a+2y)$.
57. $(a+2y)(a-2y)(a^2-2ay+4y^2)(a^2+2ay+4y^2)$.
58. $(x+1)(abx^2-abx+ab+1)$.
59. $(6a^2+3a-1)(6a^2-3a-1)$.
60. $(x-a)(x+2a+b)$.
61. $(a+b+1)(a-b)$.
62. $(a-b-c)(2a+3b+c)$.
63. $(a+b)(a-b)(a^2+b^2-1)$.

64. $2c(c^2 + 3d^2)$.
 65. $(2x-2y+1)(2x-2y-1)(x-y)$.
 66. $xy(x-y)(x-y)(x+y)$.
 67. $(a+b)(x-3)(x-2)$.
 68. $(x+y+a+b)(x+y-a-b)(a-b+x-y)(a-b-x+y)$.
 69. $(a+b)(a+b)(a+b)$.
 70. $(a-b)(a-b)(a-b)$.
 71. $(a+1)(a+1)(a+1)$.
 72. $(a-1)(a-1)(a-1)$.
 73. $(x+2)(x+2)(x+2)$.
 74. $(a^2-b^2)^3$.
 75. $(x+y+z)^2$.
 76. $(x-y+z)^2$.
 77. $(x^4y^2+3)(x^4y^2+4)$.
 78. $(a^2-ab+b^2)(a+b+1)$.
 79. $(x-y)^3$.
 80. $(2a^2+3a-1)(2a^2-3a+1)$.
 81. $(m-n-9)(m-n-3)$.
 82. $(3a^2+2ab+b^2)(3a^2+2ab+b^2)$.

Page 64.

83. $(x+4)(x+2)(x^2+5x+8)$.
 84. $(x^4+1)(x^2+1)(x+1)(x-1)$.
 85. $(3x-1)(5x^2+11)$.
 86. $(x-1)(x-2)(x-3)$.
 87. $(x^6+y)(x^2+y^2)$.
 88. $(a-3b+3)(a-3b-3)$.
 89. $9y(4x^2+2xy+y^2)$.
 90. $(b+c-5a)(b+c-a)$.
 91. $(x-b+a)(x-b-a)$.
 92. $a^2x^2(a^2+x^2)(a+x)(a-x)$.
 93. $(a+b+1)(a-1)(1-b)$.
 94. $(2x+3y)(2x-3y)^2$.
 95. $(x+2)^2(x-2)^2$.
 96. $(a^2-b^2-c^2)^2$.
 97. $(x+1)^3$.
 98. $(x+a)(x-b)$.

99. $(x-c)(x+d)$.
 100. $(4a+b+c)(a-3b+4c)$.

Page 65.

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|-----------------|----------------|
| 1. $3a^2b^2c$. | 6. $17abc$. |
| 2. $3xy^3z$. | 7. $15ax^2$. |
| 3. $6a^2x^2$. | 8. $3a^2m^2$. |
| 4. be . | 9. $4x^5$. |
| 5. a^3b . | |

Page 66.

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|---------------------------|------------------|
| 1. $5xy$. | 7. $a-b$. |
| 2. $x+y$. | 8. $a+1$. |
| 3. $x-1$. | 9. $x-1$. |
| 4. $x+1$. | 10. $(2x+3y)y$. |
| 5. $x+1$. | 11. $(x-y)^2$. |
| 6. $ab(a-b)$. | 12. $x+2y$. |
| 13. $x(5x-3y)$. | |
| 14. $(x+y)(x^2-xy+y^2)$. | |
| 15. $5x^2-1$. | |

Page 67.

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|-------------|------------|
| 1. $2x-3$. | 3. $a-1$. |
| 2. $x-a$. | 4. $x+1$. |

Page 68.

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|------------------|-----------------------|
| 5. x^2-2x+3 . | 12. $x-5$. |
| 6. $x(3x+2)$. | 13. $x-1$. |
| 7. x^2-3x+2 . | 14. x^2+x+1 . |
| 8. $x^2-13x+5$. | 15. $2x-1$. |
| 9. $x-y$. | 16. x^2-8 . |
| 10. $x+2$. | 17. $x^2(2x^3-y^2)$. |
| 11. $x(2x+3)$. | 18. $3a+b$. |
| | 1. $x-2$. |

Page 69.

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|--------------|-------------|
| 2. $2x+7$. | 5. $3x-y$. |
| 3. $x-2$. | 6. $x-5y$. |
| 4. x^2-1 . | |

Page 70.

- | | |
|--------------------|--------------------|
| 1. $210abc$. | 3. $12a^2b^2c^2$. |
| 2. $72a^4b^2c^3$. | 4. $a^2x^2y^3$. |

Page 71.

5. $12x^3y^4$.
6. $84a^2bc^2$.
7. $171x^3y^2a^2b^2c^3$.
8. $264a^4b^4c^4$.
9. $360a^3x^4$.
10. $168x^3y^2$.
1. $x^2(x-3)$.
2. $x(x+1)(x-1)$.
3. $6x(3x-1)$.
4. $x(x+1)(x-1)(x^2+x+1)$.
5. $(x+1)(x-1)(x^2-x+1)$.
6. $(x-1)(x+1)(x+2)(x^2+x-2)$.
7. $12a^2b^2(a+b)^2(a-b)$.
8. $12ax^2y^2(x^2-y^2)^2$.
9. $(a+x)(a^3-x^3)$.
10. $(a^2-ax+x^2)(a^2+ax+x^2)(a+x)$.
11. $x(1-x)^3$.

Page 72.

12. $(x^2-y^2)(x-2y)$.
13. a^6-b^6 .
14. $(x-1)^2(x+2)$.
15. $(x+y+z)(x+y-z)(x-y+z)$.
16. $5ab(a-x-1)(a+x)(a-x)^2$.
17. $(3x+1)(2x^2+3x-2)$.
18. $(2x-3y)(3x-2y)(x^2-4y^2)$.

Page 73.

1. $(3x-4)(2x+7)(4x-5)$.
2. $(2x-1)(x+2)(3x+1)$.
3. $(2x-3)(x+2)(2x-1)$.
4. $(4x+1)(2x+7)(3x-8)$.
5. $(7x-4)(3x-2)(x^2-3)$.
6. $(x+2)(x+4)(x^2+3x+1)$.
7. $(x^2-5x+6)(x-1)(x-4)$.
8. $(x^2+3x+2)(x-3)(x+5)$.
9. $(x^2+2x-3)(2x^2-3x-8)(2x^2-x-5)$.
10. $(x-2)(x-3)(x^2+2x+1)$.
11. $(2x+1)(3x^2-x-2)(3x^2+x-2)$.

12. $(x+a)^2(x^2+a^2+ax)(x^2+a^2-ax)$.
13. $(x^2-2x+3)(x-3)(x^2+2x-3)$.
14. $(3x-7)(x^2-2x+3)(2x^2+5x-3)$.
15. $(x^2-2x+4)(x^2+x-2)(x^2+6x)$.

Page 74.

1. $(x-1)(x-2)(x-3)(x-4)$.
2. $(3a+1)(a-1)(2a-1)(3a-2)$.
3. $(3x-4)(3x+4)(7x^2-5)(2x-3)$.
4. $(2x+1)(2x^2-x-1)(4x^2+x+1)(x-1)$.
5. $(x-3)(x^2-2x+3)(x^2+2x-3)(x^3+3x^2+5x+3)$.

Page 78.

1. $\frac{2}{3a^3b}$.
4. $\frac{cd}{3xy}$.
2. $\frac{2x^4}{5y}$.
5. $\frac{1}{ax-1}$.
3. $\frac{x}{2-3ax}$.
6. $\frac{4(x-y)}{1}$.
7. $\frac{x-3y}{x^2+3xy+9y^2}$.
8. $\frac{5a}{3b}$.
9. $-\frac{x}{a+x}$.
10. $\frac{x+1}{x+4}$.
1. $\frac{x+3}{x+2}$.
3. $\frac{a^2-17}{a^2-5}$.
2. $\frac{5a+7}{a-2}$.
4. $\frac{x^2-2x-2}{3x^2-6x+4}$.

Page 79.

5. $\frac{2x+3}{3x+5}$.
6. $\frac{x+7}{x+13}$.

$$7. \frac{a+b-c}{a+b+c}, \quad 8. \frac{x+1}{x-1}.$$

$$9. \frac{2x-1}{x^2+1}.$$

$$10. \frac{x^{2m} + 2x^m + 2}{x^m + 2}.$$

$$1. x-3 + \frac{4}{x}.$$

$$2. x-2 + \frac{5}{6x}.$$

3. Not reducible.

$$4. x+1 - \frac{1}{x^2}.$$

Page 80.

$$5. x+2y - \frac{y^2}{x-2y+6}.$$

$$6. x + \frac{2x}{x-1}.$$

$$7. 2x - \frac{a^2}{2a-x}.$$

$$8. 2x - \frac{1}{x-3}.$$

$$9. x^3 + x^2 + x - \frac{x+1}{x^2-x-1}.$$

$$10. x^3 + x^2 + x + 1 + \frac{2}{x-1}.$$

Page 81.

$$1. \frac{a^2-c}{y}.$$

$$4. \frac{x^2-3x}{x-2}.$$

$$2. \frac{2y^2-a^2+c}{y}.$$

$$5. -\frac{4+x^2}{x+1}.$$

$$3. \frac{x^2}{x-1}.$$

$$6. \frac{-2y^2}{x^2-y^2}.$$

$$7. \frac{x^3+x^2-4x+3}{x^2+2x-1}.$$

$$8. -\frac{2b^2}{a-b}.$$

$$9. \frac{1}{1-x}.$$

$$10. \frac{x^2-2x+1}{x^2+x-1}.$$

Page 83.

$$1. \frac{n^2-3n, 1+3n}{n-3}.$$

$$2. \frac{x+3, 2x+4}{(x+2)(x+3)}.$$

$$3. \frac{2x, 3a^2, 4ax^2}{a^3x^3}.$$

$$4. \frac{1}{x^2-1}, \frac{3x-3}{x^2-1}.$$

$$5. \frac{a+x, 1}{a^2-x^2}.$$

$$6. \frac{x(a+1)(a-1), y(a-1)z(a+1)}{(a+1)(a-1)^2}.$$

$$7. \frac{ay^2(x-y), bx^2(x+y), cxy}{x^3y^3(x^2-y^2)}.$$

$$8. \frac{(a-1)(a-2), (a+2)(a-5), (a+3)(a+3)}{(a+3)(a-5)(a-2)}.$$

$$9. \frac{a, -b}{(a-b)(b-c)}.$$

$$10. \frac{(x+1)(x^2+x+1), (x^2+1)(x-1), x^3}{(x-1)(x^2+x+1)}.$$

Page 84.

1. $\frac{x}{15}$
2. $\frac{4x}{15}$
3. $\frac{3x-25}{24}$
4. $\frac{5(2x^2-1)}{8x^2}$
5. $\frac{2-a}{1-a^2}$
6. $\frac{4x^2}{1-x^4}$
7. $\frac{a^2+2ab+b^2}{4ab}$
8. $\frac{3x-5}{(x-2)(x-1)}$
9. $\frac{2a-a^2+1}{a-1}$
10. $\frac{b}{b-c}$

Page 85.

5. 1. $\frac{6ac-f}{9c-2c^2}$
2. $\frac{3x+4}{2x^2+5x-1}$
3. $\frac{x+2}{2x-1}$
4. $\frac{x^2-17}{x^2-5}$
5. $\frac{x^{2m}+2x^m+2}{x^m+2}$
6. $\frac{x^5}{x^3-a^2y}$
7. $\frac{2x+1}{x-2}$
8. $\frac{x-3}{x^2-3x+1}$
9. $\frac{x^2+1}{x^4+x^2+1}$
10. $\frac{3x^2+x+2}{2x^2+x+3}$

Page 86.

6. 1. $a; 1 + \frac{1}{a}; 1 - \frac{1}{a};$
 $a+b - \frac{b^2}{a-b}; 2a-6 + \frac{23}{a+3}$
 $2 + \frac{a^2+x^2}{ax}$

$$2. a^2 + 2ab + 3b^2 + \frac{4b^3}{a-b};$$

$$a-1 + \frac{a-1}{a^2-2};$$

$$7x+10 + \frac{43}{x-4};$$

$$a-x + \frac{ax^2+a-x^3-x+1}{a^2+ax-1}$$

$$7. 1. \frac{2x^2-a^2}{x}$$

$$2. \frac{17x-7}{3x}$$

$$3. \frac{2a-x+1}{a}$$

$$4. \frac{10x^2+4x+3}{5x}$$

$$5. \frac{9ax-4a-7x+2}{3a-2}$$

$$6. \frac{axy+4x+5y}{x+y}$$

$$7. \frac{3x^2}{x^2-1}$$

$$8. \frac{a^3+3x^3}{a+x}$$

$$9. 2(x+1).$$

$$10. 0.$$

$$8. 1. \frac{4ab}{a^2-b^2}$$

$$2. \frac{a^2+2ab-b^2}{a^2-b^2}$$

$$3. 0.$$

Page 87.

$$4. \frac{19a-20b}{12} \quad 5. \frac{2(a^2+b)}{ab^2}$$

$$6. -a^4.$$

$$7. \frac{-4xy}{(x^2-y^2)^2}$$

$$8. \frac{a-b}{a+b}$$

$$9. -\frac{a^5}{a^6-b^6}$$

$$10. 0.$$

Page 89.

2. $b + \frac{ab}{x}$

5. $\frac{(x^2 + b^2)(x+b)(x+b)}{-a}$

3. $\frac{x^4 - b^4}{bc(b+c)}$

6. $x - 1$

4. $\frac{x-b}{6c^2x}$

7. a^2

Page 90.

8. $\frac{(acd - b^2d + bc^2)bcd}{a-b+c}$

9. $\frac{(a-x)^2}{x^3}$

11. 10.

10. $\frac{x(a+1)}{2}$

12. $+\frac{2p^3}{385qy}$

13. $\frac{x}{x^2 - xy + y^2}$

14. $\frac{1}{3ab}$

15. $\frac{y(x^2 - y^2)}{x^3}$

18. $\frac{m^2n^2}{m^2 - n^2}$

16. $1 + \frac{2x}{y}$

19. $\frac{x+1}{x+5}$

17. a

20. 1.

Page 92.

1. $\frac{d(an+m)}{n(bd-c)}$

2. $\frac{a+4c}{4x+2z}$

3. $\frac{x^2(a-3)}{a(x^2-x+1)}$

4. $\frac{5(a-b)}{10a+9b}$

5. $\frac{a(df+e)}{bdf+be+cf}$

6. $-a$

7. 1.

8. $\frac{bc+ac+ab}{a^2c+ab^2+a^2b}$

Page 93.

9. 1.

10. $\frac{-2x}{1+x^2}$

1. $\frac{2mn}{m+n}$

2. $\frac{a^2+ax+x^2}{a+x}$

3. $\frac{x+6}{x+4}$

4. $\frac{1-a}{1-a} = \frac{1}{1} = 1$

5. $\frac{6a^2-1}{2-b}$

6. $\frac{a^2+m^2}{a-m}$

1. $a^2 + 2ab + 3b^2 + \frac{4b^3}{a-b}$

2. $a-1 + \frac{a-1}{a^2-2}$

3. $7x+10 + \frac{43}{x-4}$

4. $a-x + \frac{ax^2-x^3-x+a+1}{a^2+ax-1}$

5. $x^2 + 3ax + 3a^2 + \frac{3a^3}{x-2a}$

1. $\frac{2b^3}{a^2-ab+b^2}$

2. $\frac{a^2+(b-c)^2}{2bc}$

3. $\frac{2b-a^2+b^2-c^2}{2b}$

4. $\frac{(a-1)^4(a-1)}{a}$ or $\frac{(a-1)^5}{a}$

Page 94.

5. $\frac{(x^3+y^3)^2}{x^3-3xy+y^3}$

6. $\frac{x(x^2-1)}{x^2+x+1}$

7. $\frac{x^3+x^2-2}{x+1}$

1. $\frac{b(x+y), a(x+y), a^2b^2}{ab(x^2-y^2)}$

2. $\frac{-1}{a-c}, \frac{1}{a-c}$

3. $\frac{a}{x-y}, \frac{a}{x-y}$

6. $\frac{(1-a)(ax-x^2), (a-b)(a-x), bx}{ax-x^2}$

7. $\frac{(x+2)(x+1)^2, (3x+2)(x-2)(x+1), x(2x-1)(x+2)}{x(x^2-4)(x+1)}$

1. 1. 2. -1.

3. $\frac{a^2-c^2}{a^2-b^2}$

Page 95.

4. $\frac{a}{c}$

5. $\frac{(a+b)(a+c)}{(a-b)(a-c)}$

6. $\frac{b+a}{b-a}$ 9. a

7. $\frac{xy-x^2}{y^3}$ 10. $\frac{b+1}{ab^2}$

8. $\frac{a-b-c}{a+1}$ 11. $\frac{1}{2}$

12. $\frac{yz+xz+xy}{x^2z+xy^2+y^2z}$

1. $y^3+11y^2+38y+32$

2. m^2-mn+x 4. $4a(a^2+1)$

3. $8n^4$ 5. $-\frac{1}{b}$

Page 96.

1. $x-x^2+x^3-x^4+\frac{x^5}{1+x}$

2. $1+\frac{b}{a}+\frac{b^2}{a^2}+\frac{b^3}{a^3}+\frac{b^4}{a^4-a^3b}$

3. $1+2x+2x^2+2x^3+\frac{2x^4}{1-x}$

4. $\frac{1}{1-x+x^2}, \frac{1}{1-x+x^2}$

5. $\frac{a-1}{a^3-1}, \frac{1-a}{a^3-1}$

4. $1+x-x^3-x^4+\frac{x^6}{1-x+x^2}$

5. $x-3+\frac{9}{x}-\frac{27}{x^2}-\frac{81}{x^3+3x^2}$

Page 102.

2. 1. $x=10$ 6. $x=30$
2. $x=4$ 7. $x=+2\frac{1}{2}$

3. $x=a$ 8. $x=6\frac{1}{3}$

4. $x=3$ 9. $x=10\frac{3}{7}$

5. $x=4$ 10. $x=5$

3. 1. $x=\frac{6}{7}$ 6. $x=10$

2. $x=-6$ 7. $x=5$

3. $x=-\frac{5}{3}$ 8. $x=4$

4. $x=48$ 9. $x=4$

5. $x=\frac{1}{7}$ 10. $x=10$

Page 103.

4. 1. $x=-12.5$ 4. $x=-2\frac{2}{3}$

2. $x=20$ 5. $x=6$

3. $x=12$

5. 1. $y=\frac{4a-2b}{a-3b}$

2. $x=\frac{ab-c}{3c+2a}$

3. $x=2a$

4. $x=\frac{2m}{3}$

5. $x=b-2a$

6. $x=\frac{bc-am}{a-5}$

7. $x=\frac{b(a+c)}{a+1}$

$$8. x = \frac{b(a+6d)}{3(a-4d)}.$$

$$9. x = 1 - a.$$

$$10. x = \frac{-ab}{a+b+c}.$$

$$1. x = \frac{b^2 - a^2}{2b}. \quad 4. x = 2.$$

$$2. x = -3. \quad 5. x = 0.$$

$$3. x = -\frac{6}{7}.$$

Page 104.

$$1. x = \frac{27}{9}. \quad 4. x = \frac{bc}{a+c}.$$

$$2. x = 5. \quad 5. x = \frac{1}{2}.$$

$$3. x = -1\frac{1}{3}.$$

$$1. x = 33.$$

Page 105.

$$2. x = -5. \quad 3. x = 13.$$

$$4. x = \frac{a^2bc^2 + ab + c^2}{a^2 - bc}.$$

$$5. x = 0.$$

$$1. x = 0. \quad 2. x = 4.$$

$$3. x = \frac{a(a+b)}{a-b}.$$

$$4. x = 9.$$

$$5. x = b^2 - b - 1 + \frac{1}{b}.$$

Page 106.

$$1. x = 1. \quad 4. x = -\frac{16}{11}.$$

$$2. x = 8. \quad 5. x = 4.$$

$$3. x = 4.$$

Page 107.

$$1. x = 4. \quad 4. x = -7.$$

$$2. x = 6. \quad 5. x = 2.$$

$$3. x = 13.$$

$$1. x = 2\frac{1}{2}.$$

Page 108.

$$2. x = 4\frac{1}{2}. \quad 4. x = 7.$$

$$3. x = 4. \quad 5. x = 3.$$

Page 109.

$$1. x = 6. \quad 2. x = 4.$$

$$3. x = -1\frac{1}{4}.$$

$$4. x = \frac{cd - ab}{a + b + c + d}.$$

$$5. x = -\frac{6}{7}. \quad 6. x = b + \frac{a}{2}.$$

$$7. x = \frac{c-m}{a-b-d}.$$

$$8. x = \frac{a^2 - 2ab + bc}{c - b}.$$

$$9. x = -1. \quad 11. x = -\frac{31}{8}.$$

$$10. x = a + b. \quad 12. x = 5\frac{1}{2}.$$

$$13. x = \frac{ac^3}{ab + bc - ab}.$$

$$14. x = -\frac{14}{9}. \quad 15. x = \frac{3b}{a}.$$

Page 110.

$$16. x = \frac{a}{17}. \quad 18. x = 8.$$

$$17. x = 72. \quad 19. x = 3.$$

$$20. x = \frac{bd - 2bc}{2ac - cd}.$$

REVIEW.

$$1. (2x - 5y)(x + 2y).$$

$$2. (x + y^2)(x - y^2)(x^2 + xy^2 + y^4) \\ (x^2 - xy^2 + y^4).$$

$$3. (1 + a^4)(1 + a^2)(1 + a)(1 - a).$$

$$4. (2a^m - b^n)^2.$$

$$5. (a^2 + b^2 + ab)(a^2 + b^2 - ab).$$

$$6. (x + 1)(x^2 - x + 1)(x - 1)(x^2 \\ + x + 1).$$

$$7. a^2(x - 2)^2.$$

$$8. (x - y)(x + y + 1).$$

9. $(x-y)(x+y-3xy)$.
 10. $(3a+2)(a-1)$.
 1. $5x-4y$. 4. $4a-2b-3c$.
 2. $\frac{a+x}{a}$. 5. $3x-4y-2z$.
 3. $-a$.
 1. $x=5$.

Page 111.

2. $x=a$. 4. $x=1$.
 3. $x=\frac{ac}{b}$. 5. $x=\frac{2a^3}{b-1}$.
 6. $x=2b(2b-a)$.
 7. $x=5$. 8. $x=12a^3$.
 9. $x=\frac{2bc-ac-b^2}{a-c}$.
 10. $x=\frac{1}{m}$.

Page 112.

Problems.

1. 192. 2. 250, 750.

Page 113.

3. 8, 16. 8. \$50.
 4. 100. 9. 800, 1200.
 5. 80. 10. \$120.
 6. 12, 36. 11. 28, 42.
 7. 180.
 12. Ross 20, Raymond 10.
 13. 30, 60.

Page 114.

14. 30, 12. 20. 24 feet.
 15. 1800, 800. 21. 7, 13.
 16. 12, 24, 36. 22. 20, 21.
 17. 37, 53. 23. 30, 40, 80.
 18. 18, 12; 15. 24. $3\frac{3}{7}$ days.
 19. 24 years.

Page 115.

25. \$80. 26. 100, 200, 600.
 27. 240, 120, 40, 10.
 28. 525, 600, 160.
 29. $33\frac{1}{3}$ yards. 31. 30, 18.
 30. 72 lb. 32. 72.

Page 116.

33. $5\frac{5}{11}$ minutes.
 34. $32\frac{8}{11}$ minutes.
 35. $49\frac{1}{11}$ minutes past 9.
 36. $5\frac{5}{11}$ minutes past 7.
 37. 2517, 2130, 4353.
 38. 18 miles.

Page 117.

39. $1\frac{7}{8}$ hours. 41. 24, 25.
 40. 560. 42. 144 sq. yd.
 43. $\frac{b}{1+n}, \frac{nb}{1+n}$.
 44. $\frac{ac-m}{c+d}$.
 45. 2 days. 46. $\frac{abc}{b-a}$.
 47. 300, 200, 150.

Page 118.

48. $x=\frac{abc}{b+c}$. 49. $x=\frac{ab}{a+b}$.
 50. $\frac{n(m-p)}{m-n}, \frac{m(p-n)}{m-n}$.

Page 119.

1. $x=5$. 6. $y=5$.
 $y=2$. $y=3$.
 2. $x=10$. 7. $x=4$.
 $y=4$. $y=5$.
 3. $x=8$. 8. $x=2$.
 $y=-3$. $y=-1$.
 4. $x=4$. 9. $x=3$.
 $y=10$. $y=2$.
 5. $x=-2$. 10. $x=2$.
 $y=\frac{1}{2}$. $y=3\frac{1}{5}$.

Page 120.

- | | |
|---------------------|-----------------------|
| 1. $x=3.$ | 6. $x=3.$ |
| $y=5.$ | $y=-\frac{1}{4}.$ |
| 2. $x=3.$ | 7. $x=3\frac{8}{11}.$ |
| $y=7.$ | $y=\frac{6}{11}.$ |
| 3. $x=7.$ | 8. $x=7.$ |
| $y=6.$ | $y=8.$ |
| 4. $x=\frac{1}{2}.$ | 9. $x=5.$ |
| $y=7.$ | $y=5.$ |
| 5. $x=3.$ | 10. $x=21.$ |
| $y=4.$ | $y=25.$ |

Page 121.

- | | |
|-----------|------------|
| 1. $x=7.$ | 5. $x=3.$ |
| $y=7.$ | $y=4.$ |
| 2. $x=1.$ | 6. $x=12.$ |
| $y=2.$ | $y=6.$ |
| 3. $x=1.$ | 7. $x=5.$ |
| $y=3.$ | $y=-2.$ |
| 4. $x=3.$ | 8. $x=16.$ |
| $y=-1.$ | $y=12.$ |
| 9. $x=3.$ | |
| $y=1.$ | |
10. $x = \frac{nc + bd}{mb + na}.$
 $y = \frac{mc - ad}{mb + na}.$

Miscellaneous Exercises.

- | | |
|------------|------------|
| 1. $x=20.$ | 2. $x=51.$ |
| $y=15.$ | $y=17.$ |

Page 122.

- | | |
|-------------------------------------|------------|
| 3. $x=24.$ | 5. $x=3.$ |
| $y=6.$ | $y=-4.$ |
| 4. $x=35.$ | 6. $x=10.$ |
| $y=49.$ | $y=36.$ |
| 7. $x=7\frac{1}{2}.$ | |
| $y=-2\frac{1}{2}.$ | |
| 8. $x = \frac{ab(an-bm)}{a^2-b^2}.$ | |

$$y = \frac{ab(am-bn)}{a^2-b^2}.$$

- | | |
|-------------|------------------|
| 9. $x=19.$ | 11. $x=2.$ |
| $y=3.$ | $y=\frac{1}{2}.$ |
| 10. $x=13.$ | 12. $x=ab.$ |
| $y=10.$ | $y=a+b.$ |
| 13. $x=ab.$ | |
| $y=-(a+b).$ | |
| 14. $x=4.$ | 16. $x=a(a-b).$ |
| $y=1.$ | $y=b(a-b).$ |
| 15. $x=5.$ | 17. $x=(a+b)^2.$ |
| $y=4.$ | $y=(a-b)^2.$ |

Page 124.

- | | |
|-------------------------------|----------------------------|
| 1. $x = -\frac{26}{11}.$ | 2. $x=5.$ |
| $y = \frac{13}{11}.$ | $y=3.$ |
| 3. $x = \frac{ad-bc}{dm-bn}.$ | |
| $y = \frac{bc-ad}{cm-an}.$ | |
| 4. $x = \frac{1}{3}.$ | 7. $x = \frac{2n}{1+n^2}.$ |
| $y = \frac{1}{5}.$ | $y = \frac{2n}{1-n^2}.$ |
| 5. $x=7.$ | 8. $x=6.$ |
| $y=5.$ | $y=6.$ |
| 6. $x=\frac{1}{2}.$ | 9. $x = \frac{1}{n}.$ |
| $y = \frac{1}{3}.$ | $y = \frac{1}{m}.$ |
| 10. $x = \frac{1}{n}.$ | |
| $y = \frac{1}{m}.$ | |

Page 125.

- | | |
|-----------|------------|
| 1. $x=1.$ | 3. $x=2.$ |
| $y=2.$ | $y=3.$ |
| $z=3.$ | $z=4.$ |
| 2. $x=2.$ | 4. $x=12.$ |
| $y=3.$ | $y=8.$ |
| $z=-4.$ | $z=6.$ |

Page 126.

5. $x = -2$. 8. $x = \frac{1}{8}$.
 $y = 19$. $y = \frac{1}{12}$.
 $z = 27$. $z = \frac{1}{16}$.
6. $x = \frac{1}{2}$. 9. $x = \frac{4}{3}$.
 $y = \frac{1}{3}$. $y = 4$.
 $z = \frac{1}{6}$. $z = \frac{4}{5}$.
7. $x = 8$. 10. $x = \frac{a}{2}$.
 $y = 10$. $y = \frac{b}{2}$.
 $z = 14$. $z = \frac{c}{2}$.

Page 127.

1. $y = 3$. 3. $x = -3$.
 $z = -1$. $y = 11$.
 $u = 0$. $z = 17$.
2. $x = 9$. 4. $x = 10$.
 $y = 2$. $y = 16$.
 $z = -4$. $z = 25$.
 $u = 6$.
 $w = 9$.
5. $x = \frac{b + c + d - a}{4}$.
 $y = \frac{a - b + c + d}{4}$ and so on.

Page 128.

Problems.

1. 137, 129. 2. $\frac{1}{17}$.
 3. 12 shillings.

Page 129.

4. 20, 30. 6. 20 years.
 5. 10, 15, 25. 7. $\frac{4}{15}$.
8. $x = \frac{a + b}{2}$, the greater.
 $y = \frac{a - b}{2}$, the less.

9. 63. 10. 48, 18.

11. 500, 700.

12. $28\frac{4}{17}$, $20\frac{2}{3}$, $68\frac{4}{7}$.13. $x = 4$, $y = 7$.

Page 130.

14. $x = 4$, $y = 8$.
 15. $4\frac{9}{10}$ mi., $2\frac{1}{10}$ mi.
 16. \$7400.
 17. 14,400.
 18. 12×8 .
 19. A., 54; B., $28\frac{8}{9}$; C., $21\frac{2}{3}$; D., 540.
 20. \$64,000 and \$36,000.

Page 131.

1. $x = 99$. 2. $x = 450$.
 $y = 115$ $y = 225$.
 $z = 33$. $z = 237\frac{1}{2}$.
 $u = 23$. $u = 87\frac{1}{2}$.
3. $x = 2$.
 $y = 5$.
 $z = 3$.

The number = 253.

4. 8, 5, 3.
 5. A., $\$9\frac{3}{4}$; B., $\$5\frac{1}{4}$; C., \$3.

Page 134.

1. 0°C. ; 0°R.
 2. $-23\frac{1}{3}^\circ \text{C.}$; $-18\frac{2}{3}^\circ \text{C.}$

Page 136.

1. $a^{12}c^6$. 2. $81a^8b^{12}$.
 3. $625a^4b^8c^{12}$.
 4. $-3125a^5b^{10}c^{15}$.
 5. $-343x^6y^3z^6$. 6. $729a^{18}b^{24}c^6$.
 7. $-216a^3y^{21}z^3$.
 8. $243m^{5m}b^{5c}$. 9. $4^m a^m b^{2m} c^m$.
 10. $9^n x^{6n} y^{3n} z^{15n}$.

1. $\frac{1}{32a^{10}}$
2. $\frac{243x^{20}}{32y^{15}}$
3. $\frac{256x^{24}}{6561y^8}$
4. $\frac{-x^{21}}{2187}$
5. $\frac{1}{128a^{14}}$
6. $\frac{64x^{30}}{729a^{24}}$
7. $\frac{a^8b^{12}}{c^4d^{16}}$
8. $\frac{2^m a^{2m} b^{3m} c^m}{3^m x^m y^{2m} z^{4m}}$
9. $\frac{a^{-5m} b^{10}}{x^{5m} y^{5n}}$
10. $\frac{a^{18} b^{12}}{c^{12}}$

Page 138.

1. $x^2 + 2xy + y^2$
2. $x^2 + 2x + 1$
3. $x^3 - 3x^2y + 3xy^2 - y^3$
4. $1 + 4ax + 6a^2 + 4a^3 + a^4$
5. $x^5 - 20x^4 + 160x^3 - 640x^2 + 1280x - 1024$
6. $m^8 + 8m^7n + 28m^6n^2 + 56m^5n^3 + 70m^4n^4 + 56m^3n^5 + 28m^2n^6 + 8mn^7 + n^8$
7. $m^9 - 9m^8n + 36m^7n^2 - 84m^6n^3 + 126m^5n^4 - 126m^4n^5 + 84m^3n^6 - 36m^2n^7 + 9mn^8 - n^9$
8. $(a-3)^3 = a^3 - 9a^2 + 27a - 27$
9. $(3+x)^4 = 81 + 108x + 54x^2 + 12x^3 + x^4$
10. $(y-2)^6 = y^6 - 12y^5 + 60y^4 - 160y^3 + 240y^2 - 192y + 64$
1. $a^3 - 6a^2x + 12ax^2 - 8x^3$
2. $256a^4 + 768a^3b + 864a^2b^2 + 432ab^3 + 81b^4$
3. $x^{12} + 6x^{10}y^2 + 15x^8y^4 + 20x^6y^6 + 15x^4y^8 + 6x^2y^{10} + y^{12}$
4. $32m^5 + 240m^4n + 720m^3n^2 + 1080m^2n^3 + 810mn^4 + 243n^5$
5. $2187p^7 - 10,206p^6q + 20,412p^5q^2 - 22,680p^4q^3 + 15,120p^3q^4 - 6048p^2q^5 + 1344pq^6 - 128q^7$
6. $10,000 - 400 + 4 = 9604$

Page 139.

1. $a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$
2. $x^2 + y^2 + z^2 + 2xy - 2xz - 2yz$
3. $a^2 + b^2 + c^2 - 2ab - 2ac + 2bc$
4. $l^2 + m^2 + p^2 - 2lm + 2lp - 2mp$
5. $1 + x^2 + x^4 + 2x + 2x^2 + 2x^3$
6. $a^2 + b^2 + c^2 + d^2 + 2ab + 2ac + 2ad + 2bc + 2bd + 2cd$
7. $a^2 + b^2 + c^2 + d^2 - 2ab - 2ac + 2ad + 2bc - 2bd - 2cd$
8. $x^2 + y^2 + z^2 + u^2 - 2xy + 2xz - 2xu - 2yz + 2yu - 2zu$
9. $1 + 2m + 3m^2 + 3m^4 + 4m^3 + 2m^5 + m^6$
10. $4x^6 - 12x^5 + 5x^4 + 18x^3 - 17x^2 - 6x + 9$

Page 140.

1. $a^3 - b^3 + c^3 - 3a^2b + 3a^2c + 3ab^2 + 3b^2c + 3ac^2 - 3bc^2 - 6abc$
2. $a^3 - b^3 - c^3 - 3a^2b - 3a^2c + 3ab^2 - 3b^2c + 3ac^2 - 3bc^2 + 6abc$
3. $1 - 3a + 6a^2 - 7a^3 + 6a^4 - 3a^5 + a^6$
4. $8 - 36x + 102x^2 - 171x^3 + 204x^4 - 144x^5 + 64x^6$
5. $m^6 + n^6 - o^6 + 3m^4n^2 - 3m^4o^2 + 3m^2n^4 - 3n^4o^2 + 3m^2o^4 + 3n^2o^4 - 6m^2n^2o^2$
6. $x^6 + 6x^5 + 3x^4 - 28x^3 - 9x^2 + 54x - 27$

Page 141.

1. $6b^3b^2c$
2. $-2xy^2z$
3. $9q^2r^4s^6$
4. $7l^3m^4n^6$
5. $-5mn^2x^4$
6. $4a^2b^4$
7. $3m^4n^4$
8. $-2a^nb^cm$
9. $2x^4y^3z^2$
10. $x^{n+1}y^{4m-3}$

- | | |
|------------------------------|----------------------------|
| 1. $\frac{4mn^2}{5m^4}$. | 4. $\frac{a}{3b^2c}$. |
| 2. $\frac{5ab^2c^3}{6b^3}$. | 5. $\frac{a^n}{3y^2}$. |
| 3. $-\frac{l^5m^4}{3x^3}$. | 6. $\frac{x^3y^2}{ab^2}$. |

Page 143.

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|------------------------------|-------------------------------|
| 1. $a^2 - 2a - 2$. | 3. $2x - 3y$. |
| 2. $2x^2 + 3x - 1$. | 4. $x^2 - 2x - 1$. |
| 5. $4x^2 + 3x + 10$. | |
| 6. $a - \frac{1}{2}b$. | 8. $1 + 2x + 3x^2$. |
| 7. $2a^2 + a - 2$. | 9. $2x^2 + x - \frac{1}{4}$. |
| 10. $2m + 1 - \frac{3}{m}$. | |

Page 145.

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|---------|-------------|
| 1. 73. | 6. 413. |
| 2. 232. | 7. 763. |
| 3. 785. | 8. 3546. |
| 4. 376. | 9. 5453. |
| 5. 401. | 10. 60,507. |

Page 146.

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|--------------------------------------|---|
| 1. $\frac{1}{3}$. | 9. $\frac{48}{72} = \frac{2}{3}$. |
| 2. $\frac{2}{4} = \frac{1}{2}$. | 10. $\frac{52}{65}$. |
| 3. $\frac{3}{10}$. | 11. $\frac{235}{886}$. |
| 4. $\frac{10}{100} = \frac{1}{10}$. | 12. $\frac{884}{989}$. |
| 5. $\frac{9}{13}$. | 13. $\frac{948}{967}$. |
| 6. $\frac{11}{21}$. | 14. $\frac{1000}{10000} = \frac{1}{10}$. |
| 7. $\frac{8}{15}$. | 15. $\frac{1001}{1111}$. |
| 8. $\frac{15}{27} = \frac{5}{9}$. | 16. $\frac{2222}{3333} = \frac{2}{3}$. |
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- | | |
|------------------|----------------------|
| 1. 1. .3. | 2. .9486 +. |
| 3. .12. | |
| 4. .3821 nearly. | |
| 5. .1. | 8. .5555. |
| 6. .3513 +. | 9. .0099. |
| 7. .874. | 10. .0101. |
| 2. 2. .9354. | 4. .2886 +. |
| 3. .9428. | 5. $\frac{17}{52}$. |

Page 147.

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|--------------|---------------|
| 6. .8291 +. | 9. .86602 +. |
| 7. .6928 +. | 10. 1.5545 +. |
| 8. .70716 +. | |

Page 148.

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|---------------------|----------------------|
| 2. $a + 1$. | 7. $y^2 - y - 1$. |
| 3. $x + 2$. | 8. $x^2 - 2x + 1$. |
| 4. $2m - 1$. | 9. $2x - 3x - 1$. |
| 5. $4a - 3b$. | 10. $2a^2 - a - 5$. |
| 6. $x^2 + 2x - 4$. | |

Page 150.

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|--------|--------|
| 2. 42. | 4. 32. |
| 3. 25. | 5. 47. |

Page 151.

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|----------------------|-----------------------|
| 6. 84. | 9. 526. |
| 7. 123. | 10. 632. |
| 8. 325. | 11. 1234. |
| 1. $\frac{2}{3}$. | 7. $\frac{27}{55}$. |
| 2. $\frac{1}{6}$. | 8. $\frac{23}{32}$. |
| 3. $\frac{7}{9}$. | 9. $\frac{34}{36}$. |
| 4. $\frac{4}{8}$. | 10. $\frac{41}{53}$. |
| 5. $\frac{5}{9}$. | 11. $\frac{65}{76}$. |
| 6. $\frac{12}{36}$. | 12. $\frac{82}{85}$. |
| 1. .2. | 4. .5. |
| 2. .43 +. | 5. .627 +. |
| 3. .928 +. | 6. 1.3. |

Page 152.

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|--------------|---------------|
| 7. 2.1 | 12. 3.27. |
| 8. 5.7. | 13. 3.45. |
| 9. .464 +. | 14. .01. |
| 10. 1.816 +. | 15. .00464 +. |
| 11. 1.91 +. | |
| 1. .908 +. | 6. .94 +. |
| 2. .88 +. | 7. .96 +. |
| 3. .61 +. | 8. .94 +. |
| 4. 1.07 +. | 9. .93 +. |
| 5. .87 +. | 10. .501 +. |

1. $3x + 1$. 4. $1 + x$.
2. $2x - 3y$. 5. $a - b$.

Page 153.

1. $x^{\frac{4}{3}}$. 5. $p^{\frac{1}{3}}$.
2. $y^{\frac{3}{5}}$. 6. $c^{\frac{5}{2}}$.
3. $m^{\frac{1}{2}}$. 7. $b^{\frac{4}{3}}x^{\frac{1}{5}}$.
4. $n^{\frac{1}{4}}$. 8. $y^{\frac{5}{2}}2^{\frac{4}{3}}$.
9. $3ab^{\frac{1}{m}}c^n$.
1. $\sqrt[3]{x}$. 4. $5x^4\sqrt[4]{y^3}$.
2. $\sqrt[5]{a^8}$. 5. $\sqrt[4]{a^8}\sqrt[3]{b^2}$.
3. $3\sqrt[4]{b}$. 6. $\sqrt[5]{x^3}\sqrt[3]{y^5}$.
7. $\sqrt[7]{5^4m^4}\sqrt[2]{n^7}$.
8. $a\sqrt[3]{b}\sqrt[5]{c^4}\sqrt[3]{a^7}$.
1. 32. 6. 7.
2. 243. 7. 100.
3. 9. 8. 1000.
4. 81. 9. 1.
5. 216. 10. ~ 3125 .

Page 154.

1. $\frac{a^2}{y^3}$. 6. $\frac{4b^{\frac{1}{2}}}{c^{\frac{1}{3}}}$.
2. $\frac{1}{y^5}$. 7. $\frac{3}{x^3y^{\frac{4}{9}}}$.
3. $\frac{p^3}{q^{\frac{1}{3}}}$. 8. $\frac{6}{x^5b^7}$.
4. $\frac{5x^2}{y^{\frac{1}{2}}}$. 9. $\frac{7n^7}{m^7}$.
5. $\frac{1}{a^2b^3}$. 10. $\frac{1}{x^{\frac{1}{2}}y^{\frac{3}{2}}z^{\frac{5}{4}}}$.

Page 155.

1. m^{-1} . 6. $xy^3z^{-1}u^{-\frac{1}{3}}$.
2. x^2a^{-3} . 7. $\frac{6x^3y^{-2}z^{-4}}{3}$.
3. $5a^2$. 8. $\frac{x^3a^{-\frac{3}{2}}b^{-4}}{3}$.
4. $\frac{x^{-\frac{4}{5}}}{3}$. 9. xm^5n .
5. $4a^{-3}b^2c$. 10. $m^{-\frac{4}{5}}n^{\frac{5}{7}}$.
1. $\frac{1}{3m^{-3}}$. 6. $\frac{6}{c^{\frac{1}{3}}x^6y^{-1}}$.
2. $\frac{1}{ax^{-\frac{1}{4}}}$. 7. $\frac{1}{a^{\frac{3}{5}}b^{-\frac{1}{3}}m}$.
3. $\frac{1}{x^4}$. 8. $\frac{1}{ab^{-2}c^{-3}}$.
4. $\frac{3}{7a^{\frac{4}{5}}}$. 9. $\frac{5}{7a^3b^{\frac{3}{2}}c^5}$.
5. $\frac{4}{b^{-\frac{6}{7}}}$. 10. $\frac{1}{yz^{-2}u^3}$.

Page 156.

1. a^2 . 6. 30.
2. a^{-2} or $\frac{1}{a^2}$. 7. $\frac{7}{a^4b^{\frac{1}{3}}}$.
3. $a^0 = 1$. 8. $3y^{\frac{2}{3}}$.
4. $b^{\frac{1}{2}}$. 9. $44a^{\frac{6}{5}}$.
5. $3c^{-1}$ or $\frac{3}{c}$. 10. $26xy a^{\frac{5}{3}}b^{\frac{8}{3}}$.
1. $m^{\frac{2}{3}} - n^{\frac{2}{3}}$.
2. $m^{\frac{2}{3}} + 2m^{\frac{1}{3}}n^{\frac{1}{3}} + n^{\frac{2}{3}}$.
3. $m^{\frac{2}{3}} - 2m^{\frac{1}{3}}n^{\frac{1}{3}} + n^{\frac{2}{3}}$.
4. $a^4 + a^{-4} - 2$.
5. $x^{-4} + x^{-2}y^{-2} + y^{-4}$.
6. $x + y$. 7. $x^{-6} + y^{-13}$.
8. $a + a^{\frac{1}{2}}b^{\frac{1}{2}} + b$.

Page 157.

$$9. 2x^{\frac{2}{3}}y^{-1} + 18x^{-\frac{2}{3}}y - 20.$$

$$10. 28ab^{-2} - a^{\frac{1}{2}}b^{-1} + a^{-\frac{1}{2}}b + 6a^{-1}b^2 - 28.$$

$$1. x^3a^2. \quad 6. m^{\frac{1}{3}}.$$

$$2. x^{-3} = \frac{1}{x^3}. \quad 7. a^{\frac{1}{15}}.$$

$$3. x^{-\frac{1}{45}} = \frac{1}{x^{\frac{1}{45}}}. \quad 8. \frac{4x^2}{y^{\frac{1}{4}}}.$$

$$4. x^{\frac{4}{15}}. \quad 9. \frac{2x^{\frac{1}{15}}}{a^2}.$$

$$5. \frac{5}{d^2k^{\frac{5}{8}}}. \quad 10. \frac{1}{x^{\frac{n^2+1}{n}}}.$$

Page 158.

$$12. a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}.$$

$$13. a^{\frac{3}{4}} + a^{\frac{1}{2}}b^{\frac{1}{4}} + a^{\frac{1}{4}}b^{\frac{1}{2}} + b^{\frac{3}{4}}.$$

$$14. x^{\frac{2}{3}} - 2x^{\frac{1}{3}} + 1.$$

$$15. 7x^{\frac{2}{3}} - 2x^{\frac{1}{3}} + 1.$$

$$16. x^{\frac{1}{3}} - 2 - 3x^{-\frac{1}{3}}.$$

$$17. 3x^{\frac{1}{2}} - 2 + x^{-\frac{1}{2}}.$$

$$18. a^{\frac{1}{3}} + a^{\frac{1}{6}}b^{\frac{1}{6}} + b^{\frac{1}{3}}.$$

$$19. 7a^2x + 3a^4 - 4.$$

$$20. 5b^{\frac{1}{2}} + 4b^{\frac{1}{6}} + 3b^{-\frac{1}{6}} + 2b^{-\frac{1}{2}}.$$

Squares.

$$1. a^{\frac{4}{3}} + 2a^{\frac{2}{3}}b^{\frac{1}{2}} + b.$$

$$2. a^{-\frac{2}{3}} - 2a^{-\frac{1}{3}}b^{-\frac{3}{2}} + b^{-\frac{6}{5}}.$$

$$3. 4xy + 4 + x^{-1}y^{-1}.$$

$$4. 1 - 2y^m + y^{2m}.$$

$$5. \frac{1}{m^{-2}} - 2 + m^{-2}.$$

Cubes.

$$1. a^{-2} - 3a^{-\frac{4}{3}}b^{-\frac{3}{2}} + 3a^{-\frac{2}{3}}b^{-\frac{3}{2}} - b^{-\frac{9}{2}}.$$

$$2. 8x^3 - 36 + 54x^{-3} - 27x^{-6}.$$

$$3. x^{\frac{3}{2}} - 6x^{\frac{7}{4}} + 12x^2 - 8x^{\frac{9}{4}}.$$

$$4. \frac{1}{8}x^{-\frac{3}{2}} - \frac{3}{2}x^{-\frac{3}{4}} + 6 - 8x^{\frac{3}{4}}.$$

$$5. x - 3x^{\frac{2}{3}}y^{\frac{1}{3}} + 3x^{\frac{1}{3}}y^{\frac{2}{3}} - y.$$

Values.

$$1. x^a - x^{-\frac{a}{2}} + \frac{1}{4}x^{-2a}.$$

$$2. x^{2a} - 2x^a + 4.$$

$$3. x^{4n} - 2x^{3n} + 4x^{2n} - 8x^n + 16.$$

$$4. 4x^{\frac{2}{3}} + 16x^{\frac{1}{3}} + 16 - 9x^{-\frac{2}{3}}.$$

$$5. 9x - 12x^{\frac{1}{2}} - 2 + 4x^{-\frac{1}{2}} + x^{-1}.$$

Page 159.

Transfers.

$$1. \frac{2a^{\frac{1}{3}}m^{\frac{1}{2}}c^{-1}m^{-\frac{1}{2}}}{5} = \frac{2}{5}a^{\frac{1}{3}}c^{-1}.$$

$$2. 3x^{\frac{1}{4}}y^{\frac{1}{2}}c^{\frac{1}{2}}d^{\frac{2}{3}}x^{-2}y^{-1}d^{-\frac{1}{3}} = 3x^{-\frac{7}{4}}c^{\frac{1}{2}}d^{\frac{1}{3}}.$$

$$3. (xy^2cd^{\frac{1}{4}})^{-\frac{1}{3}} = x^{-\frac{1}{3}}y^{-\frac{2}{3}}c^{-\frac{1}{3}}d^{-\frac{1}{12}}.$$

$$4. m^{\frac{1}{2}}n^{\frac{1}{3}}p^{\frac{1}{4}}a^{-\frac{1}{2}}n^{-\frac{1}{3}}p^{-\frac{1}{4}} = m^{\frac{1}{2}}a^{-\frac{1}{2}}.$$

Transfers.

$$1. \frac{4}{7a^{\frac{1}{2}}b^{\frac{3}{2}}x^{-\frac{1}{3}}y^{-\frac{2}{3}}z^{-\frac{4}{3}}}.$$

$$2. \frac{5}{d^{\frac{1}{2}}x^{-1}a^{-1}}.$$

$$3. \frac{1}{m^{-1}n^{-2}c^{-\frac{1}{3}}}.$$

$$4. \frac{5}{3b^2y^{-\frac{3}{2}}}$$

$$5. \frac{6}{7mnp^{-1}q^{-1}}$$

Operations.

$$1. a^n - b^n. \quad 5. a^{-\frac{1}{2}}b^{\frac{1}{3}}.$$

$$2. x. \quad 6. \sqrt[12]{(a+x)b^6}.$$

$$3. a^{m^2-1}. \quad 7. x^{-\frac{1}{10}}.$$

$$4. x^a. \quad 8. -a^{-2}.$$

Page 160.

$$9. -x^{-\frac{1}{2}}. \quad 12. x^{a-b}.$$

$$10. a^{mnp}. \quad 13. x^{-2ab}.$$

$$11. a^{-\frac{1}{2}}b^{\frac{1}{3}}c^{-\frac{1}{10}}.$$

$$14. (a+b)^{\frac{2}{3}} - 2(a^2-b^2)^{\frac{1}{3}} + (a-b)^{\frac{2}{3}}.$$

$$16. x^{-\frac{1}{2}} - 2y^{\frac{1}{2}} + 3x^{\frac{1}{2}}y.$$

$$17. 3a^{-\frac{3}{2}} - 5a^{-1}b - 2a^{-\frac{1}{2}}b^2.$$

$$18. a^{-\frac{1}{2}} - b^{\frac{2}{3}}.$$

$$19. x^{\frac{1}{3}} - 1 + 2x^{-\frac{1}{3}}.$$

$$20. (\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}).$$

$$21. (c^{-\frac{1}{2}} + d^{-\frac{1}{2}})(c^{-\frac{1}{2}} - d^{-\frac{1}{2}}).$$

$$22. (m^{\frac{1}{6}} + n^{\frac{1}{6}})(m^{\frac{1}{6}} - n^{\frac{1}{6}}).$$

$$23. (p^{\frac{m}{2}} + q^{\frac{n}{2}})(p^{\frac{m}{2}} - q^{\frac{n}{2}}).$$

$$24. (m^{-\frac{a}{2}} + n^{-\frac{b}{2}})(m^{-\frac{a}{2}} - n^{-\frac{b}{2}}).$$

$$25. (m^{\frac{1}{2}} + m^{\frac{1}{2}}n^{\frac{1}{2}} + n^{\frac{1}{2}})(m^{\frac{1}{2}} - m^{\frac{1}{2}}n^{\frac{1}{2}} + n^{\frac{1}{2}}).$$

Page 161.

$$26. (a^{-2} + a^{-1}b^{-1} + b^{-2})(a^{-2} - a^{-1}b^{-1} + b^{-2}).$$

$$27. (a + \sqrt{ab} + b)(a - \sqrt{ab} + b).$$

$$28. (x - 2x^{-\frac{1}{3}})(x^2 + 2x^{\frac{2}{3}} + 4x^{-\frac{2}{3}}).$$

$$29. (a^{\frac{3}{5}} + a^{\frac{3}{10}}b^{\frac{1}{10}} + b^{\frac{3}{5}})(a^{\frac{3}{5}} - a^{\frac{3}{10}}b^{\frac{1}{10}} + b^{\frac{3}{5}}).$$

Page 162.

$$1. 12\sqrt{2}. \quad 7. -3xy\sqrt[3]{4x}.$$

$$2. 7\sqrt[3]{3}. \quad 8. (a+b)\sqrt{a}.$$

$$3. 4\sqrt[3]{4}. \quad 9. x^2\sqrt[3]{y^3}.$$

$$4. 6\sqrt[3]{2}. \quad 10. 2\sqrt[3]{3}.$$

$$5. 15\sqrt[3]{6}. \quad 11. 5yz^2\sqrt[3]{2x^2z}.$$

$$6. 3ab^2\sqrt[3]{3ab}.$$

$$12. (a+b)\sqrt{a-b}.$$

$$13. (2a+3)\sqrt[3]{5}.$$

$$14. y^{\frac{p}{2}}\sqrt[3]{x^{a+p}}. \quad 20. \frac{2c}{3ab}\sqrt[3]{5b}$$

$$15. \frac{1}{2}\sqrt[3]{6}. \quad 21. \frac{3x}{a}\sqrt[3]{ax}.$$

$$16. \frac{1}{2}\sqrt[3]{2}. \quad 22. \frac{b}{2a}\sqrt[4]{a}.$$

$$17. \frac{1}{3}\sqrt[3]{3}. \quad 23. \frac{x}{y}\sqrt[3]{y^{3n-2}}.$$

$$18. \frac{1}{3}\sqrt[3]{15}. \quad 24. \sqrt[3]{2}.$$

$$19. \frac{1}{2}\sqrt[3]{77}.$$

Page 163.

$$1. \sqrt[3]{98}. \quad 7. \sqrt[3]{25x^2b}.$$

$$2. \sqrt[3]{845}. \quad 8. \sqrt[3]{\frac{3x}{a}}.$$

$$3. \sqrt[3]{81}. \quad 9. \sqrt[3]{8ax}.$$

$$4. \sqrt[3]{3645}. \quad 10. \sqrt[4]{2x}.$$

$$5. \sqrt[4]{\frac{4}{3}}. \quad 11. \sqrt{x^4-y^2}.$$

$$6. \sqrt[4]{\frac{40y}{9}}. \quad 12. \sqrt{x^2-y^2}.$$

Page 164.

1. $16\sqrt{5}$.
2. $5\sqrt{5}$.
3. $17\sqrt{7}$.
4. $-4\sqrt{7}$.
5. $8\sqrt{11}$.
11. $\frac{4}{45}\sqrt{15}$.
12. $\frac{4}{3}\sqrt{21} + 6\sqrt{3}$.
6. $-12\sqrt{11}$.
7. $-15\sqrt{3}$.
8. $9\sqrt[3]{7}$.
9. $\sqrt[3]{7}$.
10. $5\sqrt{6}$.

Page 165.

1. $30\sqrt{3}$.
2. $288\sqrt{2}$.
5. $240\sqrt[3]{4}$.
6. $\frac{2}{a}\sqrt{2}$.
7. $25\sqrt[10]{288}$.
8. $\sqrt[6]{3}$.
3. $\sqrt{x^2-4}$.
4. $2\sqrt[3]{3087}$.
9. $\frac{1}{2}\sqrt[6]{2}$.
10. $\frac{1}{40}\sqrt[3]{35}$.

Page 166.

1. $x + y - \sqrt{x+y}$.
2. $6\sqrt{21} - 46$.
3. $2 - 3\sqrt{x^2+x}$.
4. 2.
5. 0.

Division.

1. $3\sqrt{2}$.
2. $5d$.
5. $\sqrt[6]{\frac{3}{4}} = \frac{1}{2}\sqrt[6]{48}$.
6. $\sqrt[20]{\frac{8}{243}}$.
7. $\sqrt[15]{\frac{27}{2}}$.
10. $\sqrt[12]{96c}$.
3. $a^2\sqrt{3}$.
4. $\sqrt[6]{48}$.
8. $\sqrt[12]{64a}$.
9. $\sqrt[15]{\frac{9a}{2b}}$.

Page 167.

1. $6\sqrt[3]{36}$.
2. $2\sqrt[5]{54}$.
3. x^4a^2 .
4. 6.

5. $\sqrt{x} + 2y\sqrt[4]{x} + y^2$.
6. $5 + 2\sqrt{6}$.
7. $m + 2\sqrt{mn} + n$.
8. $x^3 + 3x^2\sqrt{y} + 3xy + y\sqrt{y}$.
9. $-1 - 3\sqrt[3]{2} + 3\sqrt[3]{4}$.
10. $59 + 14\sqrt{10}$.

11. $\frac{1}{2}\sqrt[3]{4}$.
12. $\sqrt{5}$.
13. $\sqrt{2}$.
14. $\sqrt[5]{3a}$.
15. $\sqrt[n]{a}$.
16. $\sqrt[3]{2}$.
17. $\frac{3a\sqrt{x}}{y}$.
18. $3x\sqrt[18]{}$.
19. $2a\sqrt[6]{8x^3(a-x)^2}$.
20. $2x\sqrt[6]{x^2(x^2-y^2)^3}$.

Page 168.

1. $\frac{1}{3}\sqrt{3}$.
2. $\frac{3}{b}\sqrt{b}$.
3. $\frac{5}{8}\sqrt[3]{2}$.
4. $\frac{x}{c}\sqrt[3]{c}$.
5. $\frac{n}{bm}\sqrt{m}$.
6. $\frac{3\sqrt[4]{1000d}}{5d}$.
7. $\frac{1}{c}\sqrt[5]{27c}$.
8. $\frac{5a}{2}\sqrt[6]{2b}$.
9. $\frac{\sqrt{1-a}}{1-a}\sqrt[3]{1-a} = \frac{\sqrt[5]{(1-a)^5}}{1-a}$.
10. $\frac{1}{a-b}\sqrt{a^2-b^2}$.

Page 169.

1. $\frac{9-3\sqrt{2}}{7}$.
2. $2(\sqrt{3}+1)$.
3. $2 + \sqrt{3}$.
5. $3 + 2\sqrt{2}$.
6. $\frac{1}{3}(4 - \sqrt{7})$.
7. $\frac{x + 2\sqrt{xy} + y}{x-y}$.
8. $\frac{m + 2\sqrt{mn-n^2}}{m-2n}$.
4. $4 - \sqrt{15}$.

$$9. \frac{3x + 2\sqrt{2x^2 + x - 1}}{x - 2}$$

$$10. \sqrt{a^4 - 1} - a^2.$$

$$1. .93 \dots \dots$$

$$2. 1.707 \dots \dots$$

$$3. 5.464 \dots \dots$$

$$4. 3.66 -$$

Page 170.

$$5. .3592 \dots \dots 7. 1.4851.$$

$$6. .101. \dots \dots 8. .608 -.$$

Page 171.

$$1. 14\sqrt{-1}.$$

$$2. 17\sqrt{-1}.$$

$$3. 19\sqrt{-1}.$$

$$4. (x + y)\sqrt{-1}.$$

$$5. 2a\sqrt{-1}. \quad 8. \sqrt{-2}.$$

$$6. 2\sqrt{-1}. \quad 9. 4m\sqrt{-3}.$$

$$7. 2x\sqrt{-1}. \quad 10. 2b\sqrt{-1}.$$

Page 172.

$$1. -30. \quad 4. -7\sqrt{-7}$$

$$2. -by. \quad 5. -abc.$$

$$3. +20\sqrt{xy}. \quad 6. -2\sqrt{-30}.$$

$$7. 14 - 10\sqrt{-2}.$$

$$8. 69 - 12\sqrt{-2}.$$

$$9. 44 + 12\sqrt{+3}.$$

$$10. 70.$$

$$1. -6\sqrt{6}. \quad 3. 50.$$

$$2. 32. \quad 4. 6\sqrt{6} - 29.$$

$$5. 14 - 8\sqrt{-15}.$$

Page 173.

$$1. 2. \quad 3. \frac{5}{2}\sqrt{-5}.$$

$$2. 2. \quad 4. \frac{2}{3}\sqrt{-15}.$$

$$5. 1 - 2\sqrt{-2}.$$

$$6. \sqrt{6}.$$

Page 175.

$$1. 2\sqrt{2} + 1. \quad 4. \sqrt{6} + 2.$$

$$2. \sqrt{2} + 1. \quad 5. 3 + \sqrt{7}.$$

$$3. 2 + \sqrt{3}. \quad 6. \sqrt{2}.$$

$$7. 2\sqrt{5} + \sqrt{6}.$$

$$8. 3\sqrt{7} + 2\sqrt{3}.$$

$$9. 4 + \sqrt{3}.$$

$$10. \frac{1}{2}(\sqrt{10} + \sqrt{6}).$$

$$11. \sqrt{7} - \sqrt{2}. \quad 12. 2 + 3\sqrt{-1}.$$

$$13. 5 + 2\sqrt{2}.$$

$$14. \sqrt{m+n} + \sqrt{m-n}.$$

$$15. \sqrt{m+x} - \sqrt{m-x}.$$

Page 176.

$$1. x=16. \quad 8. x=2.$$

$$2. x=2. \quad 9. x=6.$$

$$3. x=\frac{5}{4}. \quad 10. x=\frac{1}{4}.$$

$$4. x=25. \quad 11. x=\frac{25}{16}.$$

$$5. x=6. \quad 12. x=81.$$

$$6. x=\frac{1}{3}\sqrt{-3}. \quad 13. x=3.$$

$$7. x=-2. \quad 14. x=3.$$

Page 177.

$$15. x=100.$$

$$16. x = \frac{c\sqrt{4b-c^2}}{2}.$$

$$17. x=8. \quad 19. x=\frac{13}{4}.$$

$$18. x=\frac{16}{3}. \quad 20. x=\frac{b^2}{4a}-a.$$

Page 178.

$$1. x=\pm 6. \quad 6. x=\pm \frac{2}{3}.$$

$$2. x=\pm 3. \quad 7. x=\pm \frac{1}{1-c}$$

$$3. x=\pm 5. \quad 8. x=\pm 1.$$

$$4. x=\pm 3. \quad 9. x=\pm \sqrt{3}.$$

$$5. x=\pm 7. \quad 10. x=\pm 1.$$

11. $x = \pm \frac{a}{3} \sqrt{21}$.

12. $x = \pm \sqrt{3}$. 13. $x = \pm 4$.

14. $x = \pm \frac{1}{2}$.

15. $x = \pm \frac{b}{\sqrt{a-2b}}$.

16. $x = \frac{(a+b)^2}{2b}$.

Page 180.

1. $x=8$ or -1 .

2. $x=5\frac{1}{3}$ or -2 .

3. $x=2$ or -6 .

4. $x=10$ or -2 .

5. $x=7$ or -4 .

6. $x=2$ or -24 .

7. $x=1$ or -13 .

8. $x=(-5 \pm \sqrt{7}) 2$.

9. $x=7$ or $-\frac{19}{3}$.

10. $x=1$ or $-\frac{1}{6}$.

11. $x=5$ or -3 .

12. $x=3$ or $-\frac{14}{3}$.

13. $x=\frac{11}{5}$ or -5 .

14. $x=2$ or $-\frac{11}{6}$.

15. $x=\frac{3}{4}$ or -5 .

16. $x=\frac{3}{2}$ or $-\frac{4}{3}$.

17. $x=\frac{a}{3}$ or $-\frac{a}{5}$.

18. $x=\frac{4}{3}c$ or $-\frac{5}{4}c$.

19. $x=3$ or -1 .

20. $x=2$.

21. $x=1$ or $\frac{2}{3}$.

Page 182.

1. $x=4$ or 2 .

2. $x=14$ or -10 .

3. $x=\frac{3}{2}$ or -1 .

4. $x=\frac{5}{3}$ or $-\frac{3}{2}$.

5. $x=10$ or $-\frac{3}{2}$.

6. $x=-4$ or -1 .

7. $x=-\frac{1}{5}a$ or $\frac{1}{6}a$.

8. $x=12\frac{2}{3}$ or $-\frac{5}{3}$.

9. $x=-2 + \sqrt{5}$ or $x=-2 - \sqrt{5}$.

10. $x=m + \sqrt{1+m^2}$.

$x=m - \sqrt{1+m^2}$.

Page 183.

1. $x=1$ or $-\frac{3}{4}$.

2. $x=-2$ or $\frac{2}{3}$.

3. $x=\frac{1}{2}$ or $-\frac{3}{4}$.

4. $x=2$ or $\frac{2}{7}$.

5. $x=3$ or $-\frac{1}{5}$.

6. $x=3$ or $\frac{2}{3}$.

7. $x=4$ or $-\frac{7}{4}$.

8. $x=1$ or $-\frac{12}{5}$.

9. $x=2$ or $-\frac{3}{5}$.

10. $x=\frac{2}{3}$ or $\frac{5}{33}$.

Miscellaneous Exercises.

1. $x=3$ or $\frac{7}{6}$.

2. $x=\frac{15}{8}$ or -2 .

3. $x=\frac{2}{3}$ or $\frac{7}{4}$.

4. $x=\frac{5}{8}$ or -3 .

5. $x=-\frac{3}{8}$ or $\frac{9}{16}$.

6. $x=-\frac{7}{5}$ or $+3$.

7. $x=\frac{3}{7}a$ or $-\frac{1}{3}a$.

8. $x=-\frac{2}{3}k$ or $-\frac{5}{4}k$.

9. $x=-4$ or $-\frac{11}{6}$.

10. $x=3$ or $\frac{7}{9}$.

11. $x=6\frac{2}{7}$ or $3\frac{1}{4}$.

12. $x=9$ or 4 .

13. $x=1\frac{2}{3}$ or $-1\frac{1}{2}$.

14. $x=(a \pm b)^2$.

15. $x=(2 \pm \sqrt{5}) c$.

16. $x=5$ or $-\frac{1}{2}$.

Page 184

17. $x=1$ or $\frac{2}{3}$. 18. $x=2$ or $\frac{1}{3}$.

19. $x=10$ or $-\frac{2}{3}$.

20. $x=24$ or $\frac{42}{5}$.

Problems.

1. 25 or 5.

$$2. \frac{m \pm \sqrt{m^2 - 4n}}{2}$$

$$3. 48 \text{ or } 8, 64 \text{ or } 104.$$

Page 185.

$$4. 12 \text{ or } 8. \quad 8. 10.$$

$$5. 13. \quad 9. 120 \text{ or } 60.$$

$$6. 9 \text{ or } 45. \quad 10. \$90.$$

$$7. 9. \quad 11. 18.$$

Page 186.

$$12. 16 \text{ or } 12. \quad 16. 12 \text{ or } -15.$$

$$13. \$60 \text{ or } \$40. \quad 17. 7, 8, 9.$$

$$14. 7 \text{ or } 3. \quad 18. \$400.$$

$$15. 15 \text{ or } -12. \quad 19. 10 \text{ hours.}$$

$$20. 25.$$

Page 187.

$$21. \frac{5}{8} \text{ mi. per hour; } 4 \text{ and } 6 \text{ hr.}$$

$$22. \text{Length} = 37\frac{1}{2} \text{ ft.; breadth} = 32 \text{ ft.}$$

$$23. 9 \text{ or } 6.$$

Page 189.

$$1. x = \pm 5 \text{ or } \pm 2\sqrt{-1}.$$

$$2. x = \pm 3 \text{ or } \pm \sqrt{-1}.$$

$$3. x = 4 \text{ or } \sqrt[3]{49}.$$

$$4. x = 4 \text{ or } 1.$$

Page 190.

$$5. x = 4 \text{ or } 9. \quad 6. x = \frac{1}{4} \text{ or } \frac{1}{9}.$$

$$7. x = 5 \text{ or } 60.$$

$$8. x = 4 \text{ or } -4\frac{4}{5}.$$

$$9. x = \frac{1}{27} \text{ or } -\frac{27}{8}.$$

$$10. y = 3 \text{ or } -2; \text{ also } -3 \pm \sqrt{3}.$$

$$11. x = \frac{8}{3} \text{ or } -\frac{3}{4}.$$

$$12. x = \frac{\sqrt{-1 \pm \sqrt{1 + ab}}}{a}$$

$$13. x = \frac{1}{258} \text{ or } 16.$$

$$14. x = 6 \text{ or } -1; \text{ also, } \frac{5 \pm \sqrt{113}}{2}.$$

$$15. x = 5 \text{ or } -1; \text{ also, } 2 \pm \sqrt[3]{5}.$$

$$16. x = 6 \text{ or } -1; \text{ also, } 3 \text{ or } -2.$$

$$17. x = 2\sqrt[3]{2} \text{ or } 1.$$

$$18. x = 5 \text{ or } 2.$$

Page 193.

$$1. x = 3 \text{ or } 17. \quad 4. x = 3 \text{ or } -\frac{14}{5}.$$

$$y = 2 \text{ or } -12. \quad y = 2 \text{ or } \frac{49}{10}.$$

$$2. x = 5 \text{ or } 4. \quad 5. x = 5 \text{ or } -3.$$

$$y = 4 \text{ or } 5. \quad y = -3 \text{ or } 5.$$

$$3. x = 5 \text{ or } -3. \quad 6. x = 7 \text{ or } 1.$$

$$y = -3 \text{ or } 5. \quad y = -1 \text{ or } -7.$$

Page 194.

$$7. x = 5 \text{ or } 2. \quad 10. x = 9 \text{ or } -1.$$

$$y = 2 \text{ or } 5. \quad y = 1 \text{ or } -9.$$

$$8. x = 7 \text{ or } 3. \quad 11. x = \pm 2.$$

$$y = 3 \text{ or } 7. \quad y = \pm 1.$$

$$9. x = 3 \text{ or } 2.$$

$$y = 2 \text{ or } 3.$$

$$12. x = 2 \text{ or } -\frac{10}{9}.$$

$$y = 3 \text{ or } -\frac{29}{9}.$$

$$13. x = 60 \text{ or } -6.$$

$$y = 5 \text{ or } -5.$$

$$14. x = 36. \quad 15. x = 4 \text{ or } 2.$$

$$y = 4. \quad y = 2 \text{ or } 4.$$

$$16. x = 11 \text{ or } -13\frac{2}{3}.$$

$$y = 5 \text{ or } -7\frac{1}{3}.$$

$$17. x = 4 \text{ or } -3.$$

$$y = -3 \text{ or } 4.$$

$$18. x = 3 \text{ or } -2.$$

$$y = 2 \text{ or } -3.$$

$$19. x = a \text{ or } -a.$$

$$y = a \text{ or } -a.$$

$$20. x = 2 \text{ or } -2.$$

$$y = 5 \text{ or } -5.$$

$$21. x = 3 \text{ or } -4.$$

$$y = 4 \text{ or } -3.$$

22. $x = \frac{1}{2} (\pm \sqrt{4d + c^2} + c)$.
 $y = \frac{1}{2} (\pm \sqrt{4d + c^2} - c)$.
23. $x = 6$ or 4 . 24. $x = 9$ or 4 .
 $y = 4$ or 6 . $y = 4$ or 9 .
25. $x = \pm \sqrt{\frac{m + n^2}{2n}}$.
 $y = \pm \sqrt{\frac{m - n^2}{2n}}$.
26. $x = 2$ or -2 .
 $y = 1$ or -1 .
27. $x = -46$, or 2 .
 $y = 15$ or 3 .
28. $x = 4$
 $y = 3$
29. $x = 1$ or $-\frac{7}{11}$.
 $y = 4$ or $-\frac{11}{17}$.

Page 195.

30. $x = 2$ or $-\frac{5}{2}$.
 $y = 1$ or $-\frac{1}{2}$.
31. $x = \pm 5$ or ± 3 .
 $y = \pm 3$ or ± 4 .
32. $x = 2$ or 1 . 33. $x = \pm 3$.
 $y = 1$ or 3 . $y = 0$.
34. $x = 5, 3, 4 \pm \sqrt{-97}$.
 $y = 3, 5, 4 \pm \sqrt{-97}$.

Problems.

1. $x = 9, y = 6$.
2. $y = 10 \pm \sqrt{100 - p}$.
3. $x = 4, y = 5$.
4. $x = \frac{a}{2} \pm \sqrt{\frac{4c - a^2}{12a}}$.
 $y = \frac{a}{2} \pm \sqrt{\frac{4c - a^2}{12a}}$.
5. $x = 4, y = 7$.
6. $x = 64, y = 36$.
7. $x = 10, y = 14$.
8. $x = 15, y = 17$.

Page 196.

9. A.'s age 24, B.'s age 20.
10. 21 and 25. 15. 8 and 10.
11. $l = 33$. 16. $x = 10$.
 $b = 11$.
12. $\frac{3}{5}$ or $-\frac{1}{2}$. 17. 10, 15.
13. $x = 50$. 18. $\frac{5}{2}, \frac{5}{3}$.
14. 5 mi. per hr.

Page 197.

19. 2, 4, 8. 22. 73.
20. $x = 50$. 23. 1 in., 7 in.
21. $x = 60$ or 30 . 24. A , 15 or 5.
 $y = 40$ or 70 . B , 18 or 8.

Page 200.

1. Roots rational, real, unequal.
2. Roots rational, real, unequal.
3. Roots real and equal.
4. Roots rational, real, unequal.
5. Roots surd, real, unequal.
6. Roots rational, real, unequal.
7. Roots surd, real, unequal.
8. Roots conjugate imaginary.
9. Roots surd, real, unequal.
10. Roots surd, real, unequal.

Page 201.

1. $x^2 - 7x = -12$.
2. $x^2 + x = 6$. 3. $x^2 - \frac{3}{5}x = \frac{4}{5}$.
4. $x^2 - \frac{8}{3}x = 20$.
5. $x^2 - \frac{1}{15}x = -\frac{3}{15}$.
6. $x^2 + \frac{5}{18}x = \frac{1}{9}$.
7. $x^2 + \frac{7}{20}x = -\frac{3}{20}$.
8. $x^2 + 7\frac{1}{2}x = 0$. $[-2b^2]$
9. $x^2 - (2a + b)x = -(a^2 + ab)$
10. $x^2 - px = -\frac{p^2 - q}{4}$.

Page 202.

1. $x = -3$ or -4 .
2. $x = 2$ or -5 .

3. $x = +5$ or -4 .
 4. $x = -2$ or -6 .
 5. $x = 0, 2$, or 25 .
 6. $x = 0, 17$, or -3 .
 7. $x = 2, -1 \pm \sqrt{-3}$.
 8. $x = 1, -1, \frac{1}{2} \pm \frac{1}{2} \sqrt{-3},$
 $-\frac{1}{2} \pm \frac{1}{2} \sqrt{-3}$.
 9. $x = 1, \pm \sqrt{-1}$.
 10. $x = -\frac{2}{3}, \pm \frac{2}{3}$.

Page 208.

1. $x = 5$. 4. $x = \frac{a}{a-1}$.
 2. $x = \frac{1}{7}$. 5. $x = 12y^3$.
 3. $x = \frac{3a^4}{b}$. 6. $x = a$.

Page 209.

1. $5y^2$. 4. 136.
 2. 70. 5. $\frac{5ac-ad}{5a-b}$.
 3. $1\frac{23}{40}$. 6. $\frac{a^2+ab+b^3}{a+b}$.

1. y . 2. $4a$.

3. $\frac{a^2+2ab+b^3}{a}$.

4. $\frac{x^2-2xz+z^2}{x-y}$.

1. xy . 3. $6b^2y$.
 2. $4a^2$. 4. $9xy^2$.
 1. $a:c=d:b$. 3. $a:b=b:a$.
 2. $x^2:y^2=z^2:y$. 4. $a:a^3=d:b^3$.

2. $x = 9$.

3. $x = 3$ or $-\frac{1}{11}$.

4. $x = 3$. 5. $x = 2\sqrt{a}$.

Page 210.

6. $x = 32$. 7. $x = 17$.
 $y = 24$. $y = 11$.

8. $x = \pm a^2b$.
 $y = \pm ab^2$.

9. $x = 3$. 10. $x = \frac{5}{4}$.
 $y = 1$.

6. $x = 32$. 9. $x = \frac{3}{11} \sqrt{22}$.
 $y = 24$.

7. $x = 6$ or 2 . 10. $x = 12$.
 $y = 8$.

8. $x = 4$. 11. $x = 10$.
 $y = 10$. $y = 20$.

Page 212.

1. $l = 161$. 4. $l = -40\frac{1}{2}$.
 2. $l = 43$. 5. $l = 25x$.
 3. $l = -49$.

Page 213.

1. $l = 178, S = 1850$.
 2. $l = -41\frac{1}{2}, S = -361$.
 3. $S = 779$. 6. $S = 34,850$.
 4. $S = -483$. 7. $S = 10,000$.
 5. $S = 980\frac{1}{2}$. 8. $S = n^2$.
 9. $S = \frac{n}{2}(1+n)$.
 10. $S = \frac{a^2(3-a)}{2}$.

Page 216.

2. $a = 3, S = 595$.
 3. $l = 38, S = 258$.
 4. $a = 3$ or $-1, n = 12$ or 10 .
 5. $d = \frac{3}{7}, l = 9$.
 6. $n = 18, d = -\frac{1}{12}$.

Problems.

1. $a = 10, S = 840$.
 2. $d = .3, 2.3, 2.6, 2.9$, etc.
 3. $S = \frac{n(n+1)}{2}$.
 4. $n = 10$.

Page 217.

5. 20, 22, 24, 26.
6. 11 mi. 840 yds.

Page 218.

1. $l = 48$. 4. $l = x^p$.
2. $l = \frac{1}{2}$. 5. $l = \frac{729}{128}$.
3. $l = 1$.

Page 219.

1. $l = 2187$, $S = 3280$.
2. $l = -\frac{1}{128}$, $S = \frac{341}{128}$.
3. $l = -96$, $S = -63$.
4. $l = \frac{2187}{64}$, $S = 96\frac{33}{4}$.
5. $l = \frac{1}{6561}$, $S = 13\frac{3280}{6561}$.
6. $l = -4374$, $S = -3280$.
7. $l = \frac{81}{16}$, $S = \frac{211}{16}$.
8. $l = 3^{n-1}$, $S = \frac{3^n - 1}{2}$.
9. $l = 2510$, $S = -1705$.
10. $l = \frac{512}{\sqrt{2}}$, $S = \frac{585\sqrt{2} - 272}{2}$

Page 222.

2. $l = 256$, $S = 508$.
3. $a = 2$, $S = 254$.
4. $S = 121$, $n = 5$.
$$121 = \frac{1 \times 3^n - 1}{3 - 1} = \frac{3^n - 1}{2};$$

that is, $242 = 3^n - 1$ and
 $3^n = 243$.
But $243 = 3^1 \times 3^2 \times 3^3 \times 3^4 \times 3^5$.
 $\therefore n = 5$.
5. $l = 3^{p-1}$, $S = \frac{3^p - 1}{2}$.
6. $l = 900$, $r = 3$.
 $r^3 + 0^2 - 13r + 12 = 0$.
 $(r - 3)(r^2 + 3r - 4) = 0$.
 $\therefore r - 3 = 0$,
and $r = 3$.

8. $l = -\frac{32}{81}$, $S = -\frac{133}{81}$.
9. $l = -\frac{32}{81}$, $S = \frac{133}{81}$.

Page 223.

Infinite Series.

2. 80. 4. $\frac{27}{8}$.
3. $\frac{3}{2}$. 5. $\frac{8}{15}$.
2. 45. 3. $3\frac{1}{3}$.

Page 224.

6. 162, 54, 18.
7. -28, 14, ... $\frac{7}{8}$.
8. x^2y^2 , xy^4 .
9. $\frac{5}{8}$, $\frac{5}{4}$, $\frac{5}{2}$, 5
10. 3. 12. 2, 4, 8.
11. $1535\frac{1}{4}$. 13. $2\frac{1}{4}$.
14. $\frac{14}{15}$, $-\frac{28}{35}$, $\frac{56}{75}$
15. $\sqrt{ax}$.
16. 6, -2, $\frac{2}{3}$
17. Debt, \$4095; last pay't, \$2048.
18. Last bushel, \$196.83; total cost,
\$295.24.

Page 226.

1. $a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 +$
 $5ab^4 + b^5$.
2. $x^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 +$
 $15a^2b^4 + 6ab^5 + b^6$.
3. $a^7 - 7a^6x + 21a^5x^2 - 35a^4x^3 +$
 $35a^3x^4 - 21a^2x^5 + 7ax^6 - x^7$.
4. $a^8 + 8a^7.4 + 28a^6.4^2 + 56a^5.4^3 +$
 $70a^4.4^4 + 56a^3.4^5 + 28a^2.4^6 +$
 $8a.4^7 + 4^8 = a^8 + 32a^7 +$
 $448a^6 + 3584a^5 + 17920a^4,$
etc.
5. $x^5 - 5x^4.2y + 10x^3.(2y)^2 -$
 $10x^2.(2y)^3 + 5x.(2y)^4 + (2y)^5$
 $= x^5 - 10x^4y + 40x^3y^2 -$
 $80x^2y^3 + 80xy^4 + 32y^5$.

$$\begin{aligned}
 6. \quad & x^6 - 6x^5 \cdot \frac{3}{y} + 15x^4 \cdot \left(\frac{3}{y}\right)^2 - \\
 & 20x^3 \cdot \left(\frac{3}{y}\right)^3 + 15x^2 \cdot \left(\frac{3}{y}\right)^4 - \\
 & 6x \cdot \left(\frac{3}{y}\right)^5 + \left(\frac{3}{y}\right)^6 = x^6 - \\
 & 18x^5y^{-1} + 135x^4y^{-2} - \\
 & 54x^3y^{-3} + 1215x^2y^{-4}, \text{ etc.}
 \end{aligned}$$

$$7. \quad x^7 + 7x^6a + 21x^5a^2 + 35x^4a^3 + 35x^3a^4 + 21x^2a^5 + 7xa^6 + a^7.$$

$$\begin{aligned}
 8. \quad & (2a)^4 + 4 \cdot (2a)^3 \cdot \frac{b}{2} + 6 \cdot (2a)^2 \cdot \left(\frac{b}{2}\right)^2 \\
 & + 4 \cdot 2a \cdot \left(\frac{b}{2}\right)^3 + \left(\frac{b}{2}\right)^4 = 8a^4 + \\
 & 16a^3b + 6a^2b^2 + ab^3 + \frac{b^4}{16}.
 \end{aligned}$$

$$9. \quad 16x^4 - 96x^3y + 216x^2y^2 - 54xy^3 + 81y^4.$$

$$2. \quad -13,608x^3y^5.$$

Page 227.

$$3. \quad -\frac{330}{a^7} \quad 5. \quad 245760a^7x^3.$$

$$4. \quad 2268b^6. \quad 6. \quad 7920.$$

GENERAL REVIEW.

$$1. \quad \frac{3}{2a-b} \quad 6. \quad b.$$

$$2. \quad x. \quad 7. \quad \frac{1}{2x^2-1}.$$

$$3. \quad 4a-2b-3c. \quad 8. \quad 5x-4y.$$

$$4. \quad 2c. \quad 9. \quad \frac{a^2+ax+x^2}{a}.$$

$$5. \quad \frac{b}{a+b}.$$

Page 228.

$$10. \quad -a.$$

$$11. \quad 3x-4y-2z.$$

$$12. \quad x + \frac{1}{x} \quad 13. \quad 3a.$$

$$14. \quad 5b.$$

$$15. \quad x+3z. \quad 18. \quad \frac{a}{2x^2}.$$

$$16. \quad 4ab. \quad 19. \quad 1.$$

$$17. \quad 4a+3b. \quad 20. \quad y.$$

$$21. \quad (1) \quad (2a-b) (4a^2+2ab+b^2).$$

$$(2) \quad (a^2+b^2+ab) (a^2+b^2-ab).$$

$$(3) \quad (3m-2n) (2m+3n).$$

$$22. \quad (3x+\frac{2}{3}) (3x-\frac{2}{3}), \\ (a-b) (b-c).$$

Page 229.

$$\begin{aligned}
 23. \quad & (2c+d) (a-3b), \\
 & (n+10) (n-8), \\
 & (x+2) (x^2-2x+4) (x-2) \\
 & (x^2+2x+4), \\
 & (a+1) (a^4-a^3+a^2-a+1), \\
 & (a-b+c) (a-b-c).
 \end{aligned}$$

$$\begin{aligned}
 24. \quad & (2a+4b) (3a-b), \\
 & (x-5) (x^2+5x+25), \\
 & (1-x) (1+x) (1-x), \\
 & (x^2+x+1) (x^2-x+1) (x^4-x^2+1).
 \end{aligned}$$

$$\begin{aligned}
 25. \quad & (4+x^2+2x) (4+x^2-2x), \\
 & (3a-b) (9a^2+3ab+b^2), \\
 & (a^2+b^2) (a+b) (a-b), \\
 & (x^m+y^n) (x^{2m}+x^my^n+x^{2n}), \\
 & (2x-y) (x+2y).
 \end{aligned}$$

$$\begin{aligned}
 26. \quad & (x+y) (x^2-xy+y^2) (x-y) \\
 & (x^2+xy+y^2), \\
 & (x^2+xy+y^2) (x^2-xy+y^2) \\
 & (x^4-x^2y^2+y^4). \\
 & 2(a+2) (a-7), \\
 & (x-y) (x^4+x^3y+x^2y^2+xy^3 \\
 & +y^4), \\
 & (2a+3x) (3a-x).
 \end{aligned}$$

$$27. 3(x+5)(x-7), \\ (1+m^2+m)(1+m^2-m), \\ (2+b)(2-b)(4-2b+b^2) \\ (4+2b+b^2), \\ (a^m-b^n)(a^{2m}+a^mb^n+b^{2n}) \\ (2-y)(x+y).$$

$$28. (3x-2y)(x+y), \\ (1+x^4)(1-x^2+x^8), \\ (a-1)(a^2+a+1)(a^6+a^3+1), \\ (x+2a+b)(x+2a-b), \\ (x+y)(x^m+y^m).$$

$$29. (11a+2)(5a-1), \\ (16+x^4)(4+x^2)(2+x) \\ (2-x), \\ (a^2+a-1)(a^2-a-1), \\ (x^2+y^3)(x^4-x^2y^3+y^6), \\ (m-1)(m+1)(m-1).$$

$$30. (x^2+y^4)(x+y^2)(x-y^2), \\ a(a+2)(a-5), \\ (x-1+y)(x-1-y), \\ (a^3+b^2)(a^6-a^3b^2+b^4), \\ (x^2+x+1)(x^2-x+1)(x^4-x^2+1).$$

$$31. x=5.$$

$$32. x=2a, y=3b.$$

$$33. x=\frac{a^2+b^2}{ac-bd}, y=\frac{a^2+b^2}{ad+bc}.$$

$$34. x=\frac{2}{a+b-c}, \\ y=\frac{2}{b-a-c}, \\ z=\frac{2}{a+b+c}.$$

$$35. x=b, y=2c.$$

$$36. x=\frac{am-cdn}{a^2b+c^2d}, y=\frac{mc+abn}{a^2b+c^2d}.$$

$$37. x=\frac{bc+a}{ab^2+c}, y=\frac{a+bc}{a^2b-c^2}.$$

Page 230.

$$38. x=\frac{ds(an-bm)}{cns-bdr},$$

$$y=\frac{dns(a-b)}{adr-cms}.$$

$$39. x=9, y=3. \quad 40. x=a.$$

$$41. x=\frac{ac}{b}.$$

$$42. x=\frac{2}{a+b+c}.$$

$$y=\frac{2}{b-a+c}.$$

$$z=\frac{2}{b-a-c}.$$

$$43. x=2. \quad 48. 2x^2-3x+9. \\ y=3.$$

$$44. x-3. \quad 49. x^2-3.$$

$$45. a-3. \quad 50. 1+a^3-a^4.$$

$$46. x^2+4x+3. \quad 51. \frac{1}{2}a^{\frac{1}{2}}-\frac{1}{2}b^{\frac{1}{2}}.$$

$$47. a-1.$$

$$52. (a-b)\sqrt{b}, \frac{2}{3}\sqrt{10}, 3.$$

$$53. a^2b^2\sqrt[6]{a}, \\ \frac{7a-3b}{3}\sqrt[3]{3b}.$$

$$54. 5xy\sqrt{2x}, 3a^2b\sqrt[3]{2ab}, 2x\sqrt{3}.$$

Page 231.

$$55. \frac{z^2}{3y}\sqrt[6]{12x^3y^4z}.$$

$$56. \sqrt{b}; 8ab^2c^2\sqrt{ac}; \frac{2a}{b^2}\sqrt[5]{ab^2}; \\ \sqrt[6]{a^5}.$$

$$57. (x-y)\sqrt{x}; \sqrt{15}; b\sqrt{2a}; \\ a\sqrt{b-a}; 2ab\sqrt[3]{2ab^2-a^3}.$$

$$58. 2a\sqrt[n]{ab}; \frac{1}{(x-y)^{\frac{2}{3}}}$$

$$59. ab\sqrt{a}; mn^2; \frac{\sqrt{a}}{a^2}; a^3.$$

$$60. \frac{1}{a}\sqrt[6]{a^3b^3}; \sqrt[3]{3}.$$

$$61. x=4; y=1. \quad 63. x=3; y=1.$$

$$62. x=\frac{2}{3} \text{ or } \dots \quad 64. x=7 \text{ or } -3\frac{1}{2}.$$

$$65. x=4 \text{ or } 3; y=3 \text{ or } 4.$$

$$66. x=\frac{1}{2a} \text{ or } -\frac{1}{a}.$$

$$67. x=2; y=4. \quad 70. x=6 \text{ or } -3\frac{2}{3}$$

$$68. x=\frac{1}{2}a. \quad 71. x=3; y=\frac{1}{4}.$$

$$69. x=3; y=2.$$

Page 232.

$$72. x=3; y=2. \quad 75. x=\frac{a}{b}.$$

$$73. x=2; y=5. \quad 76. x=\frac{1}{3}; y=\frac{1}{5}.$$

$$74. x=5; y=2. \quad 77. x=7a.$$

$$78. x=7; y=3.$$

$$79. a^{4n} + a^{3n}b^n + a^{2n}b^{2n} + a^nb^{-n} \\ + a^{-n}b^n + a^{2n}b^{-2n} + a^nb^{-n}.$$

$$80. 2a^{4n} - 2x^{4n}.$$

$$81. 2a^{3n} - 2x^{3n}. \quad 82. 3a^{4n} - 3x^{4n}.$$

$$83. \frac{x^2}{a^2} + \frac{x}{a} + 1.$$

$$84. 3a^{\frac{1}{2}} - 2 - a^{-\frac{1}{2}}.$$

$$85. x(y+z) + yz.$$

$$86. x^{\frac{3}{4}} - x^{\frac{1}{2}}y^{\frac{1}{4}} + x^{\frac{1}{4}}y^{\frac{1}{2}} - y^{\frac{3}{4}}.$$

$$87. x=3.$$

$$88. x = \pm \frac{2a\sqrt{3}}{3}.$$

$$89. x=4 \text{ or } 1.$$

$$90. x=\frac{ac}{b}. \quad 92. x=9.$$

$$91. x=8\frac{1}{28}. \quad 93. x=\alpha^2 - b^2.$$

Page 233.

$$94. x=3 \text{ or } -2.$$

$$95. x = \frac{(a+b)^2}{4a}$$

$$96. 4 - 4x + x^2.$$

$$97. a - \frac{c}{z} + b.$$

$$98. 2 - 3x - x^2 + 5x^3.$$

$$99. a^2 - \frac{a}{2} + \frac{2}{a}.$$

$$100. x + x^2 + \frac{1}{x^2}.$$

$$101. 2a - a^{\frac{2}{3}} - 3a^{-\frac{1}{3}}.$$

$$102. 128a^{14} - 224a^{12}b + 168a^{10}b^2 - \\ 70a^8b^3, \text{ etc.}$$

$$103. 64a^6 - 64a^5b^2 + \frac{80a^4b^4}{3} -$$

$$\frac{160a^3b^6}{27} + \frac{20a^2b^8}{27}.$$

$$104. x^{16} - 4x^{13}a + 7x^{10}a^2 - 7x^7a^3 + \\ \frac{35x^4a^4}{8}$$

$$105. a^{16} - 8a^{14}b^3 + 28a^{12}b^6 - \\ 56a^{10}b^9.$$

$$106. 32a^5 - 60a^4b + 45a^3b^2 - \\ \frac{135a^2b^3}{8} + \frac{405ab^4}{128} - \frac{243b^5}{1024}.$$

$$107. 128a^{14} - 448a^{12}b^3 + 672a^{10}b^6 \\ - 560a^8b^9.$$

$$108. \frac{x^7}{128} - \frac{7x^6y^2}{32} + \frac{21x^5y^4}{8} - \\ \frac{35x^4y^6}{2} + 70x^3y^8. \dots\dots$$

$$109. 45x^3y^2.$$

Page 234.

$$110. \frac{20 \cdot 19 \cdot 18 \cdot 17}{1 \cdot 2 \cdot 3 \cdot 4} \cdot (3x)^{16} \cdot (4)^4.$$

$$111. 924x^{20}. \quad 112. 70x^4.$$

$$1. 80. \quad 5. 8 \text{ and } 6.$$

$$2. 60 \text{ and } 40. \quad 6. 65 \text{ and } 45.$$

$$3. 246. \quad 7. 12 \text{ and } 5.$$

$$4. 84. \quad 8. 40 \text{ and } 30.$$

$$9. \text{Sum, } 480; \text{rate, } 5.$$

Page 235.

10. 9 and 6. 11. 31 and 9.
 12. $\pm 3, \pm \frac{1}{2} \sqrt{-46}$.
 13. $\frac{prn}{100}$. 16. 17 and 18.
 14. $x=7, y=3$. 17. $x^2 + 2x - 4$.
 15. 42. 18. $\frac{10}{15}$.
 19. $x^2 - x - 1$.
 20. $P=400$; rate, = 5.
 21. 26.

Page 236.

22. 15 and 8.
 23. $3\sqrt{7} - 2\sqrt{3}$.
 25. 513. 30. $10\frac{2}{7}$.
 26. $x=3^n$ or 2^n . 31. $\frac{a^{\frac{2}{3}} - a^{\frac{1}{3}}b + b^2}{a + b^3}$.
 28. $2 - \sqrt{3}$. 34. $\frac{4}{3x + 3}$.

Page 237.

35. 2 shillings. 38. 54, 81, 108.
 36. 5 days. 39. ± 5 ; ± 3 .
 37. 4 days.
 40. At 1 P. M., 15 miles from C.
 41. 6.4 in. nearly.
 42. 4, 5, 6, 7. 43. $x = \sqrt[n]{\frac{r}{5}}$.
 44. $x^4 - 3x^3 - 4x^2 + 12x = 0$.
 45. $\frac{165a^{-\frac{17}{3}}}{4}$.
 46. $x=0$ or -5 and $\frac{1}{3}$ or $-\frac{16}{3}$.

Page 238.

48. $\frac{2^2}{3} \sqrt[4]{\frac{2}{29}}, 29 \sqrt[4]{(\frac{2}{29})^2}, 87 \sqrt[4]{(\frac{2}{29})^3}$.
 49. a^{p+m} .
 50. $6, \sqrt{-3}, -\sqrt{-3}$.
 51. $x = \frac{1}{2}$ or 0.
 $y = \frac{1}{3}$ or 0.

$$52. \frac{y(y^2 - y + 2)}{2(y + 5)} \text{ and } \frac{x - 1}{x^2}.$$

$$53. 3\text{h. } 36 \text{ min. } \quad 55. \$4.26; \$906$$

$$54. x = 5. \quad 56. 4\frac{2}{3}.$$

$$57. 2, 3, \frac{5 \pm \sqrt{37}}{2}.$$

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Page 239.

4. $a^2 - 2a + 1$. 6. $x = 23,000$.
 5. 64 and 36. 7. $x = 11,360$.

Page 240.

8. $x=4, y=5$.
 9. $x=12$. 10. $x=8, y=5$.
 12. $2a - b$.
 13. $6m^2 + 3bc - x$.
 14. $x + 1$. 15. $\frac{c(3a - b)}{a(a + x)}$.
 16. $x=7$.
 $y=3$.
 17. (1) $x=5$ or $6\frac{1}{2}$.
 (2) $x=4a$ or $-a$.
 18. (a) 4 in Algebra.
 12 in Arithmetic.
 48 in Grammar.
 (b) $x=500$.
 $y=700$.

Page 241.

19. (a) $x=50$. (b) $x=29$.
 $y=60$. $y=19$.
 $z=11$.

$$20. \frac{3b^{\frac{4}{3}}x}{2a^{\frac{19}{6}}}$$

$$22. \frac{2n^2(n^2 - z^2)}{m^2(n + 2)}$$

$$23. x = \frac{c}{3}.$$

$$24. \frac{a-n}{1+m}, \frac{ma+n}{1+m}.$$

$$25. \frac{1}{9} \text{ day.}$$

$$26. (1) x-y=1.$$

$$(2) x^2-y^2=x+y.$$

$$27. ac^2x^n z^{\frac{3}{m}n}. \quad 28. x = \frac{a}{3} \sqrt[3]{3}.$$

Page 242.

$$29. x=6.$$

$$y=8.$$

$$30. \frac{(a^2-x)x^2y^2}{b^3(c+1)}.$$

$$31. \frac{c-m}{a-b-d}. \quad 35. x=64.$$

$$y=8.$$

$$32. x=73.$$

$$y=27.$$

$$33. A., 500,$$

$$B., 400,$$

$$C., 300.$$

$$38. \frac{1}{3}.$$

$$34. B., 100,$$

$$A., 300.$$

Page 243.

$$39. a. (2a-5x)(2a-5x).$$

$$b. (ab+1)(ab-1).$$

$$c. (x+5)(x+3).$$

$$40. x=68.$$

$$43. x=3.$$

$$y=2.$$

$$41. 2x^2-x+1. \quad 44. \frac{c-m}{a-b-d}.$$

$$42. x=16 \text{ mi.}$$

$$45. x=5.$$

$$46. (a) (x-y)(x^2+xy+y^2) \\ (x+y)(x^2-xy+y^2).$$

$$(b) (a+b+4)(a+b-4).$$

$$(c) (a^2+b^2)(a^2+b^2).$$

$$(d) (g-22)(g+5).$$

$$(e) (x+12)(x-8).$$

$$47. x=30.$$

$$y=36.$$

$$48. x=2.$$

$$y=8.$$

$$z=3.$$

$$49. 49\frac{1}{11} \text{ min. past 9.}$$

Page 244.

$$50. x=\sqrt{-11}. \quad 52. x=27.$$

$$51. 60 \text{ and } 48.$$

$$53. x=4 \text{ or } 3.$$

$$y=3 \text{ or } 4.$$

$$54. 266 \text{ miles.} \quad 55. 1, 6, 11.$$

$$57. y=60, x=40.$$

$$58. 246.$$

Page 245.

$$60. 243a^{10} - 40\frac{5}{2}a^8b + 1\frac{3}{2}a^6b^2 -$$

$$4\frac{5}{4}a^4b^3 + 1\frac{5}{16}a^2b^4 - \frac{b^5}{32}.$$

$$61. x=80.$$

$$64. \frac{1}{a}.$$

$$63. 2 - \sqrt{3}.$$

$$65. (b) x=10\frac{1}{2}.$$

$$66. x=2, y=4, z=6.$$

$$67. x=80.$$

$$68. x = \frac{a+bc}{ab^2+c}.$$

$$69. x=6.$$

$$70. x=3.$$

Page 246.

$$72. 5x-4y.$$

$$73. a+x+\frac{x^2}{a}.$$

$$74. x = \frac{am-cdn}{a^2b+c^2d}.$$

$$y = \frac{cm+abn}{c^2d+a^2b}.$$

$$75. x = \frac{a}{2} \text{ or } -\frac{5a}{4}.$$

76. (a) $(2x-5y)(x+2y)$.
 (b) $(x+y^2)(x^2-xy^2+y^4)$
 $(x-y^2)(x^2+xy^2+y^4)$.
 (c) $(1+a^4)(1+a^2)(1+a)$
 $(1-a)$.
 (d) $(2a^m-b^n)(2a^m-b^n)$.
 (e) $(a^2+b^2+ab)(a^2+b^2-ab)$.

77. (a) $\sqrt{a^2-b^2}$.
 (b) $\sqrt[4]{2b}$.
 (c) $15bc\sqrt{2a^2c}$.
 (d) $\frac{9a}{2x}\sqrt{3}$.

78. $2-3x-x^2+5x^3$.

79. $\frac{x+2}{x^2+3x+3}$.

80. $x=12$.
 $y=5$.

82. b . 83. $\frac{1}{2x^2-1}$.

84. $x = \frac{b(c^2-2b^2)}{c^2-2b^2}$.

Page 247.

85. (a) $x = \frac{1}{a}$ or $-\frac{1}{2a}$.
 86. (a) $(4+x^2+2x)(4+x^2-2x)$.
 (b) $(3a-b)(9a^2+3ab+b^2)$.
 (c) $(a^2+b^2)(a+b)(a-b)$.
 (d) $(x^m+y^n)(x^{2m}-x^my^n+y^{2n})$.
 (e) $(2x-y)(x+2y)$.

87. (a) $\sqrt{b}$.
 (b) $4abc\sqrt[3]{2bc^2}$.
 (c) $\frac{2a}{b^2}\sqrt[5]{ab^3}$.
 (d) $\sqrt[6]{a^5}$.

88. $\frac{a}{2} + b - \frac{c}{2}$

89. $\frac{a^5}{32} - \frac{5a^4}{8b} + \frac{5a^3}{b^2} - \frac{20a^2}{b^3} + \frac{40a}{b^4}$
 $-\frac{32}{b^5}$.

90. $\frac{2x-1}{x^2+x+1}$.

91. $x=8, y=6$.

92. $x=3$ or -2 .

93. $x=2, y=4$.

94. 65 and 45.

95. $3x-4y-2z$.

96. $a^2 - \frac{a}{2} + \frac{2}{a}$.

97. Sum = 480, rate = 5.

Page 248.

98. $\frac{x^7}{2^7} - 7 \cdot \frac{x^6}{2^6} \cdot 2y^2 + 21 \cdot \frac{x^5}{2^5} \cdot (2y^2)^2 -$
 $35 \cdot \frac{x^4}{2^4} \cdot (2y^2)^3 - 35 \cdot \frac{x^3}{2^3} \cdot (2y^2)^4$.

99. $x = \frac{(a+b)^2}{2b}$.

100. $x=3, y=\frac{1}{2}$.

101. (a) $(x-y)\sqrt{x}$.

(b) $\sqrt{15}$.

(c) $b\sqrt{2a}$.

(d) $a\sqrt{b-a}$.

(e) $2ab\sqrt[3]{2ab^2-a^3}$.

102. $x^{\frac{3}{4}} - x^{\frac{1}{2}}y^{\frac{1}{4}} + x^{\frac{1}{4}}y^{\frac{1}{2}} - y^{\frac{3}{4}}$.

103. $x=9$ or -6 .

$y=6$ or -9 .

105. $\frac{o \times e \times 144}{n \times w}$.

106. $a^4 - 2a^2b^2 + 4abc^2 + b^4 - c^4$.

107. (a) $(2m-x)(n-y)$.

(b) $(a+14)(a-4)$.

(c) $7a(2b-1)(2b-1)$.

$$108. (a) a - \frac{5}{a+m}.$$

$$(b) -\frac{1+a^4}{1-a^4}.$$

Page 249.

$$110. (a) z \pm 5\frac{5}{9}.$$

$$(b) x = \frac{1}{2}, y = \frac{1}{4}.$$

$$111. x = 120.$$

$$112. 7225, 5625. \quad 113. 9 - m + n.$$

$$114. (a) 2a^3b^2c\sqrt{6cn}.$$

$$(b) \frac{1}{3}\sqrt{15}.$$

$$(c) 7\sqrt{3}.$$

$$(d) \frac{a^4}{b^2}.$$

$$115. mp - (ab + cd) = s.$$

$$116. x^5 + 8x^2y^2 - 8x^2y^3 + 16xy^4 - 16xy^3 - 32y^5.$$

$$117. (a) (x-9)(x-1).$$

$$(b) (2a+b)(c+d).$$

$$(c) abc(ab-bc+a^2c^2).$$

$$118. \frac{1}{(x-1)(x+2)}.$$

Page 250.

$$119. x = a; y = b.$$

$$120. x = 160. \quad 122. x = \pm 21.$$

$$y = \pm 6.$$

$$121. x = 2. \quad 123. 2x^2 - x + 1.$$

$$y = 1\frac{1}{2}.$$

$$z = 1.$$

$$124. (a) \frac{1}{3}\sqrt{35}.$$

$$(b) \frac{5}{a}\sqrt{x}.$$

$$(c) 81a^2b^8.$$

$$125. (a) a^4(a-7)(a-5).$$

$$(b) (2y+1)(2y-1)(2y+1)(2y-1).$$

$$(c) (x-a)(x^2+ax+a^2).$$

$$(d) (x+2)(x+1)(x-1).$$

$$126. x = 24.$$

Page 251.

$$127. x = 3, y = 2.$$

$$128. 20, 24.$$

$$129. (a) \sqrt[6]{a^5}.$$

$$(b) 3m^4\sqrt{6am}.$$

$$(c) \sqrt{a+b}.$$

$$(d) \frac{81}{x^2y^{-4n}}.$$

UNIVERSITY OF PENNSYLVANIA.

$$130. (a) (p+x-2)(p-x+2).$$

$$(b) (a+b)(a-b)(a+b)(a+b).$$

$$131. (a) x-3.$$

$$(b) x^2-2x+3.$$

$$132. \frac{1}{1+x^2}.$$

$$133. \frac{ab}{a-b}. \quad 134. x = 7, y = 4.$$

Page 252.

$$135. 60 \text{ yds.}$$

$$136. (a) (x-1)\sqrt{-1}.$$

$$(b) 9 + \frac{5}{2}\sqrt{10}.$$

$$137. (a) \frac{243b^{10}}{32a^{10}x^4}.$$

$$(b) \sqrt[3]{a^2} - \frac{1}{x^4}.$$

$$138. (a) x = 6 \text{ or } -1.$$

$$(b) x = 3a \text{ or } 2b.$$

$$(c) x = 81 \text{ or } 1.$$

$$139. (a) -\frac{b}{a}.$$

$$(b) x^2 - 4x = 21.$$

$$(c) x^2 - 2x = 6.$$

140. $x=7, y=5$.

141. $x^5 + 5x^2 + 10x^{-1} + 10x^{-4} + 5x^{-7} + x^{-10}$.

142. $1 - \frac{x}{3} - \frac{x^2}{2}$.

Page 253.

143. $6x^2 - 7x^{\frac{5}{3}} - 19x^{\frac{4}{3}} + 5x + 9x^{\frac{2}{3}} - 2x^{\frac{1}{3}}$.

144. $x = \frac{(a-2)^2}{4}$.

145. $\frac{x^{\frac{4}{3}} + 2x^{\frac{2}{3}}}{x-1}$.

146. $ab^{\frac{3}{2}}x - \frac{1}{2}y^{\frac{1}{2}}$.

147. Any two digits one of which is twice the other will satisfy the conditions of the problem.

148. $y = 6 \pm \sqrt{30}$.

$$x = \frac{6}{6 \pm \sqrt{30}}$$

149. 2, 4, 6, 8.

150. $a^3 + b^3 + c^3$

151. (a) $[5(a+b) + 6c][2(a+b) - c]$.
 (b) $(a+b-1)^2$.

Page 254.

152. (a) $x - 2a$.

(b) $x^2 + 2x - 3$.

153. $-\frac{a^2+1}{a^3-1}$ 154. 0.

155. $x=1, y=2, z=3$.

156. 5, 10.

157. $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If $a = 0$, then $x = \frac{-b \pm \sqrt{b^2}}{0}$

$$= \frac{-b \pm b}{0} = \frac{0}{0}, \text{ or } \frac{-2b}{0} =$$

Infinity.

158. (a) $x = a$ or $-\frac{1}{a}$.

(b) $x = 5$ or $-\frac{1}{2}$.

159. $\sqrt[3]{a^3}, \frac{1}{\sqrt[6]{a^2}} = \frac{1}{\sqrt[3]{a}}, (a^{\frac{1}{2}})^5 = \sqrt[10]{a^5}$.

160. $x = 1$ or -2 .

$y = 2$ or -1 .

161. $\frac{(\sqrt{2} + \sqrt{3} - \sqrt{5})\sqrt{6}}{12}$.

162. $16a^4 - 96a^5 + 216a^6 - 216a^7 + 81a^8$.

163. $S = 36\frac{3}{4}$.

Page 255.

164. 1. 165. $ab(a-b)$.

166. $(a^2 - 3a + 2)(a^2 + 2a + 4)(a^2 + 3a - 1)$.

167. m .

168. $16\frac{4}{11}$ min. past 12 o'clock.

169. $x=4, y=1$. 171. x^{a^2} .

170. 25 or $\frac{1}{4}$. 172. $y^{\frac{2}{3}\frac{5}{2}}$.

174. $16a^2 - 32a^{\frac{5}{2}}b^{\frac{2}{3}} + 24a^3b^{\frac{4}{3}} - 8a^{\frac{7}{2}}b^2 + b^{\frac{5}{3}}a^4$.

175. $a=5, S=5$.

176. $(x-3)(x^2+3x+9)(x-12)(x^2-2)$.

177. $\frac{(2x+1)(x-2)(x+1)}{2x-1}$.

Page 256.

$$178. \frac{-2a^2c}{a-c} \quad 179. \frac{1}{(x+1)^2}$$

$$180. \frac{x^2-3xy+y^2}{x^2-y^2}$$

$$181. x=4. \quad 182. 1504.$$

$$183. x=\frac{1}{5}, y=\frac{1}{3}, z=1.$$

$$184. (a) \frac{2(a+bx)}{(a^2-b^2)(1-x^2)}$$

$$(b) 0.$$

$$185. (a) (a+b)(a+b)(a-b).$$

$$(b) (x+1)(x-1)([px+qx+p-q]).$$

$$(c) (a^2-3c^2)(a^2-4b^2+c^2).$$

Page 257.

$$186. (a) x=2\frac{4}{7}, y=2\frac{3}{4}.$$

$$(b) x = -\frac{2bc}{b^2+a^2}$$

$$y = \frac{2ac}{b^2+a^2}$$

$$187. x = \frac{a-b}{2}.$$

$$188. -3x. \quad 190. x^{16}-a^{16}.$$

$$191. (a) (2x-3y)(x+4y),$$

$$(b) (a+1)(a-b-1),$$

$$(c) (x-1)(x-3)(x-1).$$

$$192. L. C. Dd. = (a+b)(a-b)(a^2-ab+b^2)(a^2+ab+b^2).$$

$$193. (a) \frac{2x}{1+x}$$

$$(b) \frac{b^5}{ac}$$

$$194. 1. \quad 195. 0.$$

Page 258.

$$196. 9\frac{1}{2}.$$

$$197. x = \frac{ab+c}{b+1}, y = \frac{a-c}{b+1}$$

$$198. x-4,$$

$$y = -1.$$

$$199. 64a^3 - \dots \dots \dots$$

$$200. a^{xy}.$$

$$201. \sqrt{x^4-1}-x^2.$$

$$202. 9\sqrt{3}.$$

$$203. x=17 \text{ or } -1.$$

$$204. x=3, y=1.$$

$$205. (a) \frac{2(\sqrt{2}+\sqrt{x})}{\sqrt{1-x}}$$

$$(b) .3942$$

$$206. \frac{x^{2m}+x^my^4+y^8}{x^m+y^4}$$

Page 259.

$$207. a^{\frac{5}{2}}-5a^{\frac{3}{2}}+10a^2 \dots \dots \dots$$

$$208. 0.$$

$$209. \frac{(\sqrt{2}-2\sqrt{3}+\sqrt{5})(9+4\sqrt{6})}{-15}$$

$$210. x=4 \text{ or } 1. \quad 212. m=3x-1$$

$$211. x=2 \text{ or } 1.$$

$$y=1 \text{ or } 2. \quad 213. x=7, y=8.$$

$$214. 2x=24,000.$$

$$215. (a) x = \frac{7 \pm 3\sqrt{5}}{2}$$

$$(b) x = \frac{b}{2a}$$

$$216. x^2-2x+13=0.$$

Page 260.

$$217. 2-\sqrt{3}. \quad 218. 2024^m.$$

$$219. (a) a^4b^{10}c^4.$$

$$(b) \frac{\sqrt[3]{x^2y^{21}}}{x^2y^{21}}$$

$$220. \sqrt{3}-\sqrt{2}.$$

$$221. \frac{(7+2\sqrt{6})(9+3\sqrt{6})}{81-54}$$

$$222. 5.$$

$$223. x=17 \text{ or } 3. \quad 224. l=21.$$

$$S=121.$$

HAVERFORD COLLEGE.

225. (a) $y = -5$.

(b) $y = \frac{1-3z}{z-1}$.

226. (a) $(x - \frac{1}{4})(x + \frac{5}{4})$.

(b) $(a-b)(x-y)$.

Page 261.

227. (a) $x = 13$.

(b) $x = 6, y = 6$.

(c) $x = \frac{11}{3}$ or 1.

228. 65.

229. $x = \sqrt{3} - \sqrt{2}$.

230. $(x+y)(x^2-xy+y^2)(x-y)$
 (x^2+xy+y^2) .

231. $\frac{-2}{1+x^2}$.

232. $x = 3\frac{1}{2}, y = 5\frac{1}{4}$.

233. a or $\frac{1}{a}$.

234. 40.

Page 262.

235. $x = 2\sqrt{-3}$. 236. 12 miles.

$y = \sqrt{-3}$.

LAFAYETTE COLLEGE.

240. $(x+y)(x-y)(x^2+xy+y^2)$.

241. $(x-8)(x-7)$.

242. $x = 13$.

243. $-15x^{\frac{11}{5}}y^{\frac{5}{6}}$.

244. $\sqrt[6]{a}\sqrt[15]{b^2}$, or $\sqrt[30]{a^5b^4}$.

245. $x = 4$.

246. $x = 3$ or 25 .

$y = 2$ or -9 .

247. $x = 2$ or $-\sqrt[3]{5}$.

Page 263.

248. $x = 3$. 249. $x = 30$.

$y = 5$.

$z = 7$.

250. $x = \frac{mb+a}{mn-1}$.

$y = \frac{an+b}{mn-1}$.

BRYN MAWR COLLEGE.

252. $2x^2 + 18$.

253. I. $(x^2+1)(x+1)(x-1)$
 $(x+1)(x^2-x+1)$.

II. $(a^2+b^2)(a^2+c^2)(b+c)$
 $(b-c)$.

254. I. $x = \frac{2}{3}$.

II. $x = 9$.

255. $x = 36$ or 20 ; $y = 20$ or 36 .

Page 264.

257. $x = .00638 \dots$.

259. (a) $\frac{1}{64}$.

(b) 50.

(c) 44.362.

261. $ax^3-bx^2+bx \dots$.

262. I. $x = -\frac{65}{3}$.

II. $x = -184, y = -367$.

264. H. C. D. $= a^2 + xy$.

Page 265.

265. I. $2x + 1$.

II. 4.

III. $2(a+b+c)$.

266. \$667.95. 267. $x = 7, y = 4$.

268. $x^2 + 4x = +1$.

270. 4, 12, 36.

271. $a + 4a^{\frac{1}{2}} + 6 + \frac{4}{a^{\frac{1}{2}}} + \frac{1}{a}$.

272. \$300.

Page 266.

273. I. $x = 4\frac{1}{2}$.

II. $x = 1, y = 1$.

274. $\frac{2(a^2+b^2+a^2b-b^3)}{a^4-2a^2b^2+b^4}$.

275. 0.

278. I. 10,201.

II. 1.

280. $\frac{2b}{\sqrt{a-b}}$.

281. $\frac{ac-b^2}{a-2b+c}$.

Page 267.

282. 4.5 very nearly.

STATE COLLEGE.

283. (a) $(x-5y)(x-4y)$.
 (b) $(a+b)(a^2-ab+b^2+1)$.
 (c) $(x^2-a+1)(x^2-a-1)$.
 (d) $2(b-a)(13a^2+22ab+13b^2)$.

284. $x=4$ or 2 , $y=2$ or 4 .

285. $x = \frac{1 \pm \sqrt{-89}}{6}$.

286. $(x-1)(x^2+x+1)=0$.

287. $-3.146 \dots$

288. $x = \frac{b'c-bc'}{ab'-a'b}$.

289. $16x^2 - 32x^{\frac{1}{2}} + 24x^{-1} - 8x^{-\frac{5}{2}} + x^{-4}$.

290. $x=3$ or -1 .

Page 268.

PRINCETON.

291. (a) $(x+y)(x-y)(y-3x)$.
 (b) $(y+c-x)(y-c+x)$.
 (c) $(9a^2+4b^2)(3a+2b)(3a-2b)$.

292. (a) $2a^{\frac{1}{2}}c$.

(b) $\frac{1}{x^a}$.

(c) $\frac{\sqrt{7}-\sqrt{2}}{5}$.

293. $2x^{\frac{n}{2}} - 4 + 3x^{\frac{n}{2}}$.

294. (a) $x = \frac{pr-q^2}{p^2-qr}$; $y = \frac{r^2-pq}{p^2-qr}$.

(b) $x=5$; $y=4$.

295. In the root we have $\sqrt{p^2-4q}$,
 and when the quantity
 under the radical sign is
 negative, the expression is
 imaginary.

298. 54.

299. (a) $2(5a-5b+1)(25a^2-50ab+25b^2-5a+5b+1)$.

(b) $(1+ax-by)(1-ax+by)$.

(c) $(x^2-y^2+4xy)(x^2-y^2-4xy)$.

Page 269.

300. (a) $a^n(n-1) + a$.

(b) $5\sqrt{5} + 11$.

(c) $2^{n^2-2n} + 2^{2n}$.

301. $5x^{\frac{2}{3}} - 3x^{\frac{1}{3}} + 4$.

302. (a) $x = \frac{bc}{a^2+b^2}$; $y = \frac{ac}{a^2+b^2}$.

(b) $x=1$, $y=2$.

303. $\pm \frac{\alpha}{3} \sqrt{21}$.

304. $x=11$, $y=5$.

307. First mean $= -7\frac{1}{3}$.

308. 1.

310. $x^3 - 8x^5 + 28x^4 - 56x^3 + 70$.

311. (a) $4x^2y(7x-5)(x+3)$.

(b) $(1+x+y)(1-x-y)$.

(c) $(a+b)(a^2-ab+b^2+1)$.

Page 270.

312. $a^{4n(p-q)}$.

313. $3x^{\frac{1}{2}} - 2 + x^{-\frac{1}{2}}$.

314. (a) $x = \frac{ap}{q}$.

$$y = \frac{br}{p}$$

(b) $x = \frac{1}{3}$ or $-\frac{2}{3}$.

$$y = \frac{2}{3}$$
 or $-\frac{1}{3}$.

315. $k = -\frac{1}{2}$.

316. $x = 12$; $y = 3$.

318. $r = \frac{1}{2}$; means, 80, 40, 20, 10.

319. 15.

320. $32a^5x^{10} - 80a^4x^{11} + 80a^3x^{12} - 40a^2x^{13}$.

321. $p^{\frac{2}{5}} + p^{\frac{3}{10}}q^{\frac{1}{5}} + p^{\frac{1}{5}}q^{\frac{2}{5}} + p^{\frac{1}{10}}q^{\frac{3}{5}} + q^{\frac{4}{5}}$.

322. $2x^2 - 3x + 1$.

323. $\frac{y^4}{x^4}$. 324. $x = 3, y = 4$.

Page 271.

325. 19,851.

326. (a) $x = \frac{-m \pm \sqrt{m^2 - 4ln}}{2l}$.

(b) $x = 3\frac{2}{3}$ or $-2\frac{4}{3}$.

327. $x = 1$ or $-\frac{1}{7}$.

$$y = 6$$
 or -2 .

328. 4305.

329. $1 - x + \frac{1}{6}x^2 + \frac{1}{54}x^3 + \frac{1}{216}x^4$.

330. 15 $(3x - y)(x + y)$.

331. $\sqrt{a} - \frac{1}{b}$.

332. $n = \frac{1}{30}$. 333. $x + \sqrt{a}$.

334. 30.5308 . . .

335. $-2\frac{4}{7}$. 336. $\frac{2 \pm 2\sqrt{29}}{7}$.

337. $x = 7$.

$$y = 4$$
.

Page 272.

338. $(x^2 - y^2)(x^2 - y^2)(x^3 - y^3)$
 $(x^2 - xy + y^2)$.

339. 0. 340. $\frac{a^{21}}{b^{21}}$.

341. $x = 144, y = 96$.

342. 15.921.

343. $x = \frac{-7 \pm \sqrt{35}}{2}$.

344. $x = \pm 13$ or ± 1 .

$$y = \pm 1$$
 or ± 13 .

345. $2\frac{2}{3}\frac{3}{9}\frac{3}{3}\frac{3}{2}\frac{5}{16}$.

AMHERST.

346. (a) $(3x - 2)(x - 5)$.

(b) $(2x + 3)(4x^2 - 6x + 9)$.

347. $y = 2 \pm \sqrt{3}$.

348. $x = 5, y = -3$.

Page 273.

349. $a^{\frac{2}{3}} - a^{-\frac{2}{3}}$.

350. $x = \$6000$. 351. $S = -8\frac{5}{16}$.

352. $ax^2 + 2cx + b = 0$.

354. $4a - 4b$.

355. L. C. Dd. $= x(x^2 - 9)(x - 2)^3$.

356. x . 357. 25 and 15.

358. $\sqrt{12} + \sqrt{5}$.

359. $x = -\frac{2}{3}$ or -1 .

Page 274.

360. $2\frac{1}{4}$. 362. $7x^2$.

361. $x = 3, y = 1$. 363. 180.

YALE COLLEGE.

364. $\frac{1}{2}$.

365. (a) $(a - 2b^5)(a^2 + 2ab^5 + 4b^{10})$.

(b) $(9x^2 + y^4z^6)(3x + y^2z^3)$
 $(3x - y^2z^3)$.

$$366. \frac{3\sqrt{1-\frac{1}{2}\sqrt{2}} + \frac{1}{3}\sqrt{5} - \frac{1}{6}\sqrt{10}}{2}$$

$$367. x=2, y=7. \quad 368. C^{\frac{13}{2}}.$$

Page 275.

$$369. 10 \text{ quarters.} \\ 20 \text{ half dollars.}$$

$$374. \frac{(2\sqrt{3} - \sqrt{-2} + \sqrt{6})(8 + 4\sqrt{3})}{16}$$

$$375. \frac{3}{12}\sqrt{6}. \quad 376. x = \frac{3}{4}a.$$

$$377. x = 10 \text{ miles.}$$

$$378. x = -2. \\ y = -6.$$

Page 276.

$$379. x = 8 \text{ or } -5.$$

$$380. x = \pm 2\sqrt{3}.$$

$$381. 120 \text{ feet.}$$

$$382. 120. \quad 383. \frac{2}{3}, \frac{4}{3}.$$

MOUNT HOLYOKE.

$$384. (a) ([a+b]^2 + 9c^2)(a+b+3c)(a+b-3c). \\ (b) a(a+3b)(a-2b). \\ (c) (b+1)(b-1)(a+1)(a-1). \\ (d) (a+1)^3(a-1)^3.$$

$$385. x = \frac{2a}{a+c}, y = \frac{2b}{b-c}, \\ z = \frac{2c}{a-b}.$$

Page 277.

$$386. 2\sqrt{6} + \frac{2}{3}\sqrt{54}.$$

$$387. (a) x=1. \\ y=5. \\ (b) x=5^a \text{ or } (-1)^a.$$

$$388. (a) 5b^2. \\ (b) x=25; y=16.$$

$$370. x=6 \text{ or } 2 \text{ or } 4.$$

$$371. \frac{5}{8} + 1 + \frac{5}{8} + \frac{9}{25} \dots \dots \dots$$

$$372. \$3 \text{ per dozen and } \$6 \text{ per dozen.}$$

$$373. (a) (3x^3+1)(1+2y)(1-2y).$$

$$(b) 3(\sqrt{a}-\sqrt{b})^2.$$

$$(c) (\sqrt{5}+\sqrt{3})^2.$$

$$389. \text{ I. } x = \frac{c \pm \sqrt{c^2 - ab}}{2a}.$$

$$\text{II. } c^2 = ab.$$

$$\text{III. } ab > c^2.$$

$$\text{IV. } x^2 - x = \frac{7}{4}.$$

$$390. (a) 3\sqrt{7} - 2\sqrt{3}.$$

$$(b) \frac{a^2}{16} - \frac{a^2 b^2}{3} + \dots \dots \dots$$

$$391. (a) \frac{2y^4 + x^4}{(x^2 + xy + y^2)(x-y)^2 y^2}. \\ (b) x - \frac{a+b}{b}.$$

$$393. \frac{\sqrt{3} + 3\sqrt{-1}}{-4}.$$

Page 278.

$$394. (a) x=5 \text{ or } -7.$$

$$(b) x=9 \text{ or } 4. \\ y=4 \text{ or } 9.$$

$$395. 3\frac{197}{43}; 3\frac{2}{3}.$$

$$396. (a) 64\sqrt{-1} + 96\sqrt{-5}, \text{ etc.} \\ (b) x^2 - y^2.$$

$$397. (a) 6 - 6\sqrt{6}. \\ (b) \frac{b^2}{ac^4}.$$

$$398. 9 \text{ or } 12.$$

$$399. (a) x = \frac{1}{2}.$$

$$(b) x = \frac{1}{2}. \\ y = -\frac{1}{2}.$$

Page 279.

JOHNS HOPKINS.

401. $L. C. Dd = (a^2 + 1)(a^2 - 1).$

402. (a) 0.

(b) $\frac{4}{3x + 3}$

403. (a) $(x-2)(ax^2-3).$

(b) $(a+b+4)(a-b+4).$

(c) $x^n + y^n$ is divisible by $x + y$ if n be any odd whole number.

404. $1 - 2x^{\frac{1}{3}} + 3x^{\frac{2}{3}} - 4x.$

405. 15 gal.

406. (a) $x = \frac{-3 \pm \sqrt{-23}}{4}.$

(b) $x = 2$ or $-1.$

407. (a) $5 - \sqrt{2}.$

(b) $-\sqrt{-1}.$

(c) $\frac{2(\sqrt{2} - \sqrt{3} + 1)(1 - \sqrt{3})}{1}.$

408. $x = 3$ or $2.$

$y = 2$ or $3.$

Page 280.

409. (a) $x = 16$ or $\frac{1}{16}.$

(b) $x = 4a - 4,$ or $0.$

410. $a = \frac{ar}{1-r}$

$a - ar = ar$

$2ar = a$

$r = \frac{a}{2a} = \frac{1}{2}.$

VASSAR COLLEGE.

412. $x = 3, y = 1.$

414. $x = \frac{8}{3}.$

416. $1 + 7x^2 + 21x^4 + 35x^6 + 21x^{10} + 7x^{12} + x^{14}.$

417. $4n - 1.$

418. $6x^2 - 13x = -6.$

419. $x = 15.$

Page 281.

421. Denominator = 2.

422. $x = \frac{1}{2}$ or $-4\frac{1}{6}.$

OHIO.

Examination for Teacher's Certificate.

423. (a) $x = 9$ or $-(5^{\frac{2}{3}} + 5).$

(b) $x = 7$ or $-\frac{2}{5}.$

424. $x^{\frac{1}{5}} - 7x^{\frac{1}{5}}y^{-\frac{1}{2}} + \dots$

425. $x = 13\frac{1}{2}, y = 4\frac{1}{2}.$

426. $x = 6, y = 4.$

427. 7.

428. (a) $(a+1)(a+1)(a-1)(a-1)(a^2+1).$

(b) $(x+y-z)(x-y+z)(x+y+z)(x-y-z).$

(c) $(x-6)(x+1)(x-4)(x-1).$

Page 282.

430. (a) 0.

(b) $\frac{8}{1-x^3}$

431. $x = 5.$

432. (a) $4a^4b(1-4b^2-3b+2b^2).$

(b) $(3x+2)(2x^2-3).$

(c) $(d+e)(c-f)(c-f).$

(d) $(x^2+x+1)(x^3-x+1).$

(e) $(a-9)(a+5).$

433. $x = 8.$

$y = 18.$

$z = 16.$

434. (a) $-\frac{1}{3}\sqrt{2}$.

(b) $12\sqrt[6]{\frac{(a+x)^5}{a-x}}$.

435. $x^{10} - 2x^9 + 4x^8 - \dots$

436. (a) $\frac{b^2}{a^2x^3y^5}$.

(b) $2^{\frac{4}{3}}a^{\frac{2}{3}}x$.

437. (a) $x=42$ or 5 .

(b) $x=5$ or -2 .

438. 72 sheep.

Page 283.

439. \$36, \$32, \$27.

440. (a) $-4a\sqrt{-1}$.

(b) $13y\sqrt{3}$.

441. (a) $(z^2 + a)(x + a)(x - a)$.

(b) $(a - b)(a - b + 2c)3$.

(c) $(7b^2 + 9bc - 5c^2)(7b^2 - 9bc - 5c^2)$.

442. $\frac{1}{abc}$. 443. 4 days.

444. $7\sqrt[3]{3x} - 3x\sqrt[3]{3x}$.

445. $x=\frac{1}{3}$ or 27 .

446. $x=2$ or 3 ; also, $-3 \pm \frac{1}{2}\sqrt{2}$.

447. $x=4$ or 6 .

$y=6$ or 4 .

448. 8 miles per hour.

Page 284.

VIRGINIA.

449. (b) $(y - z)^2(y^2 + xy + z^2)^2$.

450. (a) $4a - c$.

(b) $4(a + x)(a - x)$.

451. $\frac{10cx + a - b}{10cx - a + b}$.

452. $x=10$.

453. 8 days.

454. $x=12, y=8, z=6$.

455. (a) $a + 2\sqrt{ac} + c$.

(b) $\sqrt{\frac{bx}{ay}}$.

456. $x=6$.

457. $x=3$ or $-4\frac{2}{3}$.

458. $x=3, y=6$.

Page 285.

WHEELING, WEST VIRGINIA.

High School Examination.

463. $1 - 5x + x^2$.

464. $x - \frac{2}{x}$. 465. $2\sqrt{-1}$.

466. $x=\frac{5}{2}$ or -2 .

467. A., 36; B., 20.

468. 140 days.

Page 286.

470. $2x + 1$.

472. $x = \frac{b(c-y)}{d}$.

474. $\sqrt[21]{\frac{a^7x^{11}}{y^3}}$.

475. $\frac{x(x^2 - y)}{3(x - y)}$. 477. $6\frac{2}{3}$ hours.

476. $x=8$. 478. $x-7$.

Page 287.

480. $4x - 5a$.

481. $9a^4y^2n^2 + 12a^2c^3xyn^2 + 4c^6x^2n^2$.

482. $(3c + 2a - 3b)(3c - 2a + 3b)$.

483. $3(3x + 5y)(x - y)$.

484. $\frac{m^2 - 2mn - 2n^2}{m - n}$.

485. \$4000.

486. $\frac{4}{21}$.

487. A., 1000; B., 600; C., 100.

488. A. 13, B. 7, C. 4.

Page 288.

490. $x=2$. $y=-1$. $z=1$.491. $x=2a^2(a+b+c)+abc$, $y=ab+ac+bc$, $z=a+b+c$.

492. 24 days.

493. $x=7$ or $1\frac{1}{2}$.494. $x=\frac{a^2+b^2}{a+b}$.

495. 25 and 30 miles per hour.

496. 256 ft. 497. 12 ft.

Page 289.

INDIANA.

500. $x=28, y=18$.501. (a) $\sqrt[4]{\frac{b}{a}}$.(b) $\sqrt[6]{a}$.502. $x^ny^{\frac{1}{m}}$. 504. $5\sqrt{-1}$.505. $x=64; y=36$.507. $x=4, y=9, z=16, u=25$.

Page 290.

NORTH CAROLINA.

509. 0.

510. $y^4-12y+36$. $16x^4-40x^2y^2+25y^4$. $a^2b^2-4ab^2c+4c^4$.511. $x^2+10x-56$. x^2+7x-8 . $x^2-9x+20$.512. $y=7\frac{3}{5}$. $y=-12\frac{2}{5}$.

513. A., 13; B., 11.

514. (a) $x-1$.(b) $x+y$.(c) $m+n$.515. $x=18, y=12$.

Page 291.

516. $x=-13\frac{1}{3}, y=-9\frac{2}{3}, z=-8\frac{2}{3}$.517. $x=8$.

NEW HAMPSHIRE.

519. $\frac{1}{x^3}; a^2-b^2; x^3-3x^2y+3xy^2$. $-y^3; a-b+c-d+e+f;$ $\sqrt{\frac{a}{3}} \text{ or } \frac{1}{3}\sqrt{3a}$.521. (a) $(a^2+b^2)(a+b)(a-b)$.(b) $(b+y)(a+x)$.

522. 80 inches.

Page 292.

SOUTH DAKOTA.

523. $\frac{2a(a+1)^2}{a-1}$.525. $x=3, y=7, z=4$.

526. \$70,000.

527. $\sqrt[4]{a^6}, \sqrt[4]{(5b)^4}, \sqrt[4]{(a^2+b)^3}$.528. Saltpetre, 1; sulphur, 1;
charcoal, 8.

KENTUCKY.

529. $(x^2-xy+y^2)(x^2+xy+y^2)$
 $(x+y)(x-y),$ $(y-1)(y^2+xy+1),$ $(x+1)(x^4-x^3y+x^2y^2-xy^3$
 $+y^4).$ 530. $2x^2+7x+3$.531. $\frac{1}{2x^2-1}$.

Page 293.

532. $x = 3$.
 533. 47 sheep.
 534. $5\frac{5}{8}$ miles per hour.
 535. (a) $\sqrt{2}$.
 (b) $49 + 20\sqrt{6}$.
 (c) $x = \frac{2}{3}$.
 536. $x = 9\frac{1}{7}$ or -11 .
 537. $66\frac{2}{3}$ miles.
 538. 16 miles.

Page 294.

COLORADO.

539. $(x^2 + xy + y^2)(x^2 - xy + y^2)$.
 540. $\alpha^0 = \frac{\alpha}{\alpha} (= \alpha^{1-1}) = 1$.

$\sqrt[3]{a}$ ($= \alpha^{\frac{1}{3}}$) means the cube root of a .

$\alpha^{\frac{2}{5}}$ ($= \sqrt[5]{a^2}$) means the 5th root of the square of a .

α^{-n} ($= \sqrt[n]{\frac{1}{a}}$) means the n th root of the reciprocal of a .

541. $x = \pm 2$ or $\pm \frac{1}{2}\sqrt{-10}$.
 542. $x = 1$ or $\frac{1}{13}$.
 543. $S = 5\frac{509}{12}$.
 544. Assign numerical values and simplify.

NASHVILLE, TENN.

545. (a) -8640 .
 (b) $8 - 5 - 3 = 8 - 8 = 0$.

Page 295.

546. (a) $2bx + 2by$.
 (b) $2x$.
 547. 55 and 20.
 548. (a) $a^3 - 3abc + b^3 + c^3$.
 (b) $9x^2 - 49$.
 (c) $16x^4 - 16x^3 + 4x^2$.

$$549. 6a^{n+1} - 5a^n b + 4a^{n-1}b^n + a^{n-1}b^2 - 2a^{n-2}b^{n+1}.$$

$$550. x = 2.$$

$$551. m^2 + 2mn + mp + n^2 + np + p^2.$$

$$552. x = a - b.$$

$$553. a^n + 2a^{m+1} - 4a^{m+2} + 5a^{m+3}.$$

554. A child, \$2.
 A woman, \$6.
 A man, \$6.

$$555. (a) (a + d + 2b - 3c)(a + d - 2b + 3c).$$

$$(b) 2(3x + 2y)(2x - 3y).$$

$$556. 2x^3 - 4x^2 + x - 1.$$

$$557. \frac{m(m-n)}{m+n}.$$

$$558. 14 \text{ miles.}$$

Page 296.

$$559. x = a, y = b, z = c.$$

$$560. 759.$$

$$561. (a) \frac{a^n x^2}{b^x y^n}$$

$$(b) a^{\frac{1}{n^2}} b^{-n} x^{-3}.$$

$$562. 2^3 - 8.2^{\frac{x}{a}} + 28.2^{\frac{x}{a}} \left(\frac{x}{a}\right)^2 \dots$$

$$563. x^2 + x - \frac{1}{3}.$$

$$564. x^2 - ax - a^2.$$

$$565. x^{-2} + 2x^{-1} - \frac{1}{3}x^{\frac{1}{3}}.$$

$$566. x = a + 2b \text{ or } a - b.$$

$$567. x = 4 \text{ or } 3.$$

$$568. 150 \text{ ft.}$$

Page 297.

$$570. 3 - \sqrt{-2}.$$

$$571. (a) \frac{15 \times 14}{1 \times 2} = 105 \text{ ballots.}$$

$$(b) 6 \times 5 \times 4 \times 3 = 360.$$

$$572. \frac{35x^4y^6}{1944}$$

$$574. x=5 \text{ or } 4.
x=4 \text{ or } 5.$$

$$575. r=\frac{2}{3}, n=\infty.$$

$$576. \frac{+\sqrt{-15} + \sqrt{15} - 3\sqrt{-1} - 5}{2}$$

$$577. \frac{1}{2}(\sqrt{6} - \sqrt{2}).$$

$$578. x=0, y=1.$$

$$579. 495.$$

$$580. x = \frac{-1 \pm \sqrt{5}}{2}$$

Page 298.

NEBRASKA.

$$581. 1 + 2x. \quad 582. x = \frac{mnt}{m+n}$$

$$583. x = 9 \pm 2\sqrt{10}.$$

$$585. (a) (x-7)(x-1).$$

$$(b) (x-5)(x+4).$$

$$(c) (x+12)(x-3).$$

$$(d) (x+y+z)(x-y-z).$$

$$586. x=2, y=3, z=4.$$

$$587. x=9, y=4.$$

$$588. 2x^2 - 7.$$

$$589. x=5.$$

$$590. a^{2m} + b^{2m}.$$

Page 299.

UTAH.

$$591. 2a^2(a-3b)(2a+3b)(2a-b)(3a-2b).$$

$$592. \frac{bm-an}{m-a} \text{ and } \frac{100(a-b)}{bm-an}$$

$$593. (a) x = -2 \text{ or } -2\frac{3}{8}.$$

$$(b) x=0 \text{ or } -2; \text{ or } -1 \pm 2\sqrt{-1}.$$

$$594. x=2a \text{ or } -a.$$

$$y=2b \text{ or } -b.$$

$$595. x=150 \text{ or } 100.$$

$$y=100 \text{ or } 150.$$

$$596. -6 - 13\sqrt{-2}; 7\sqrt{2}.$$

$$597. \frac{9}{35}; \frac{25}{27}.$$

$$598. 3, 5, 7, 9.$$

Page 300.

$$599. (a) \frac{1}{2x^2-1}.$$

$$(b) \frac{x-1}{x^3}.$$

$$600. (a) (x+2y)^2(x-y)(x^2-2xy+4y^2).$$

$$(b) 3(m-3)(4m+5)(4m^2+m-3).$$

$$601. (a) x+3.$$

$$(b) a+2a+4.$$

$$602. (a) \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4}$$

$$x^{n-4}y^4.$$

$$(b) a^7 + 7a^6b + 21a^5b^2 \dots$$

$$603. x = (m-n+1)m.$$

$$y = -m(m-n-1).$$

$$605. \frac{x^{\frac{1}{2}}y^{\frac{19}{6}}}{z^3}.$$

$$607. \$2.25; \$1.75.$$

$$608. x=50.$$

Date Due

25 Apr '31 F

1196

18 Dec '46 G

10637



3 1198 03949 8725



N/1198/03949/8725X

371.3M

R 198.2

Rawlins

Linnacott's elementary

371.3M

R198.2

3 1198 03949 8725



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